Statistics 341 Name ______________________________
Exam #3
November 14, 2008

INSTRUCTIONS: Read the questions carefully and completely. Answer each question and show all your work in the space provided. Credit cannot be given if work is not shown. Good luck!

1. (26 pts) The probability density function for a random variable Y is defined as

$$f(y) = \begin{cases} y & 0 \le y \le 1 \\ 0.25 & 1 < y \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) (10 pts) Find $F(y)$ for the random variable Y.
   The real line is divided by the density function into 4 regions. For $y < 0$, $F(y) = 0$. For $y > 3$, $F(y) = 1$.
   For $0 \le y \le 1$,

$$\begin{aligned} F(y) &= \int_{-\infty}^{0} f(y)dy + \int_{0}^{y} tdt \\ &= 0 + \frac{t^2}{2}|_0^y \\ &= \frac{y^2}{2} - 0 \\ &= \frac{y^2}{2} \end{aligned}$$

   For $1 < y \le 3$,

$$\begin{aligned} F(y) &= \int_{-\infty}^{0} f(y)dy + \int_{0}^{1} ydy + \int_{1}^{y} 0.25dt \\ &= 0 + \frac{y^2}{2}|_0^1 + 0.25t|_1^y \\ &= \frac{1}{2} - 0 + 0.25y - 0.25 \\ &= 0.25y + 0.25 = 0.25(y+1) \end{aligned}$$

   Putting it all together, we have

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{y^2}{2} & 0 \le y \le 1 \\ 0.25(y+1) & 1 < y \le 3 \\ 1 & y > 3 \end{cases}$$

(b) (6 pts) Find $P(0.5 \le Y \le 1.5)$.

$$\begin{aligned}
P(0.5 \le Y \le 1.5) &= P(Y \le 1.5) - P(Y \le 0.5) \\
&= F(1.5) - F(0.5) \\
&= 0.25(1.5+1) - \frac{0.5^2}{2} \\
&= 0.625 - 0.125 \\
&= 0.5
\end{aligned}$$

(c) (10 pts) The expected value of the random variable Y is $\frac{4}{3}$. Find the variance of the random variable Y.

$$\begin{aligned}
E(Y^2) &= \int_0^1 y^2 y dy + \int_1^3 0.25y^2 dy \\
&= \int_0^1 y^3 dy + \int_1^3 0.25y^2 dy \\
&= \frac{y^4}{4}|_0^1 + \frac{y^3}{12}|_1^3 \\
&= \frac{1}{4}(1-0) + \frac{1}{12}(27-1) \\
&= \frac{1}{4} + \frac{26}{12} \\
&= \frac{29}{12} = 2.4167
\end{aligned}$$

$$\begin{aligned}
V(Y) &= E(Y^2) - (E(Y))^2 \\
&= \frac{29}{12} - \left(\frac{4}{3}\right)^2 \\
&= 0.6389
\end{aligned}$$

2. (10 pts) Let the random variable Y have a uniform distribution with minimum value $\theta_1 = -a$ and maximum value $\theta_2 = a$, where a is some constant. Show that

$$E(Y^r) = \begin{cases} 0 & \text{if r is a positive odd integer} \\ \frac{2a^r}{r+1} & \text{if r is a positive even integer} \end{cases}$$

The first step is to set up the expected value.

$$\begin{aligned} E(Y^r) &= \int_{-a}^{a} \frac{y^r}{2a} dy \\ &= \frac{1}{2a(r+1)} y^{r+1}|_{-a}^{a} \\ &= \frac{a^{r+1} - (-a)^{r+1}}{2a(r+1)} \end{aligned}$$

The last line above will evaluate differently if r is even or if r is odd. If r is odd, $r+1$ is even, and $(-a)^{r+1} = a^{r+1}$. This gives

$$\frac{a^{r+1} - (-a)^{r+1}}{2a(r+1)} = \frac{a^{r+1} - (a)^{r+1}}{2a(r+1)} = 0$$

If r is even, $r+1$ is odd and $(-a)^{r+1} = -(a)^{r+1}$. This gives

$$\frac{a^{r+1} - (-a)^{r+1}}{2a(r+1)} = \frac{a^{r+1} + (a)^{r+1}}{2a(r+1)} = \frac{2a^{r+1}}{2a(r+1)} = \frac{a^r}{r+1}$$

3. (10 pts) The random variable Y has the following density curve.

$$f(y) = \begin{cases} \frac{y}{8} & 0 \leq y \leq 4 \\ \\ 0 & \text{elsewhere} \end{cases}$$

Describe how you would generate 10000 values in R from this distribution. I should be able to take your answer and generate the data in R myself.

The first step is to find the $F(y)$ function.

$$F(y) = \int_{-\infty}^{y} f(t)dt = \int_{-\infty}^{0} 0dt + \int_{0}^{y} t/8dt = 0 + t^2/16|_0^y = y^2/16$$

The values of $F(y)$ are the probabilities values for a given value of y. To generate observed probabilities, we always use the uniform distribution with $\theta_1 = 0$ and $\theta_2 = 1$. The code in R is

```
u<- runif(10000,0,1)
```

We then need to transform these probabilities into values of y using the $F(y)$ function. Since $u = y^2/16$, $y = 4\sqrt{u}$. The R code is

```
y<- 4*sqrt(u)
```

4. (10 pts) Suppose the weight of M&M bags has a normal distribution with mean 50 grams and standard deviation 2 grams.

   (a) (3 pts) What percent of M&M bags will weigh between 49.0 grams and 52.5 grams?
   For this problem, we need $P(49 \leq Y \leq 52.5)$. This is the same as $P(Y \leq 52.5) - P(Y \leq 49)$. You can calculate this probability in R with the command

   ```
   pnorm(52.5, 50, 2) - pnorm(49, 50, 2)
   ```

   The probability is 0.5858.

   (b) (3 pts) The labeled weight of the M&M bags is not the same as the mean of 50 grams. What should the labeled weight of the package of M&Ms be so that only 10% of the bags are underfilled (less than the labeled weight)?
   We would like to find the value of y so that $P(Y \leq y) = 0.1$. We can find this value using R with the command

   ```
   qnorm(0.1, 50, 2)
   ```

   The value of y is 47.44 grams.

   (c) (4 pts) Now assume the mean weight μ is unknown, but the standard deviation is still $\sigma = 2$ grams. What should the mean weight of the packages of M&Ms be so that only 10% of the bags are underfilled when the label weight is 47.9 grams?
   We have from the problem that $P(Y \leq 47.9) = 0.1$. From this equation, we need to determine the value of μ.

$$P(Y \leq 47.9) = P\left(Z \leq \frac{47.9 - \mu}{2}\right) = 0.1$$

   This means that $\frac{47.9-\mu}{2}$ =qnorm(0.1, 0, 1)
   Solving for μ gives $\mu = 47.9 - 2*$qnorm(0.1,0,1) = 50.4631 grams.

5. (14 pts) A manufacturing plant uses a specific bulk product. The amount of product used in one day can be modeled by the gamma distribution with $\alpha = 4$ and $\beta = 2.9$ (measurements in tons).

(a) (3 pts) What is the probability the plant will use more than 12 tons of this bulk product in any given day?

This is $P(Y > 12) = 1 - P(Y \leq 12)$. We can determine this probability in R with the command

```
1 - pgamma(12, shape = 4, scale = 2.9)
```

The probability is 0.4070

(b) (3 pts) How much of this bulk product should be stocked so that the plant's chances of running out of the product on any given day is only 0.005?

For this problem, we need to find the value of y so that $P(Y > y) = 0.005$. This implies that $P(Y \leq y) = 0.995$. We can find this value using the R command

```
qgamma(0.995, shape = 4, scale = 2.9)
```

This value of y is 31.8349 tons.

(c) (8 pts) The cost for this bulk product is \$90 per ton. What is the expected daily cost and the variance of the daily cost for this product?

First we need to determine the expected amount of the product used and the variance of this amount.

$$
\begin{aligned}
E(Y) &= \alpha\beta = 4(2.9) = 11.6 \\
V(Y) &= \alpha\beta^2 = 4(2.9)^2 = 33.64
\end{aligned}
$$

Then we can determine the expected value and variance of the cost function.

$$
\begin{aligned}
E(C) &= E(90Y) = 90E(Y) = 90(11.6) = 1044 \\
V(C) &= V(90Y) = 90^2V(Y) = 90^2(33.64) = 272484
\end{aligned}
$$