Stat 544 – Spring 2005

Homework assignment 4 answer key

Exercise 1

(a) Let X be the number of simulations needed to produce one observation from the truncated normal. So $X \sim$ Geometric(p), where p is the acceptance rate, i.e. the probability of obtaining one observation $\geq a$. So $p = 1 - \Phi(a)$. Note that Range(X) = {1,2,...}. Hence, the average number of simulations is $\text{E}(X) = [1 - \Phi(a)]^{-1}$

(b) Note that a has been replaced by μ for the sake of clarity

b.1 - Upper bound

$$\left.\begin{array}{rcl} f(x) & = & e^{-x^2/2} \\ g(x) & = & \alpha e^{-\alpha(x-\mu)} \end{array}\right\} \Rightarrow h(x) = \frac{1}{\alpha} e^{\alpha(x-\mu)} e^{-x^2/2}$$

$$\frac{\partial}{\partial x} h(x) = \alpha e^{-x^2/2} e^{\alpha^2 - \alpha\mu}$$

Thus, $h(x)$ attains its maximum when $x = \alpha$. Hence, the upper bound ignoring truncation is

$$\frac{1}{\alpha} e^{\alpha^2/2 - \alpha\mu}$$

Now, if we introduce the truncation we only need to consider values of α such that $\alpha \geq \mu$. So, we consider two cases for the bound:

i. $\alpha > \mu$: Upper bound is $\frac{1}{\alpha} e^{\alpha^2/2 - \alpha\mu} := K(\alpha)$

ii. $\alpha = \mu$: Upper bound is $\frac{1}{\alpha} e^{-\alpha^2/2} = \frac{1}{\mu} e^{-\mu^2/2}$

b.2 - Now need to find α that minimizes upper bound in each case

i.

$$\frac{\partial}{\partial \alpha} K(\alpha) = K'(\alpha) = -\frac{1}{\alpha^2} e^{\alpha^2/2 - \alpha\mu} + \frac{1}{\alpha} e^{\alpha^2/2 - \alpha\mu}(\alpha - \mu) = \frac{e^{\alpha^2/2 - \alpha\mu}}{\alpha^2}(\alpha^2 - \alpha\mu - 1)$$

$$K'(\alpha) = 0 \Leftrightarrow \alpha^2 - \alpha\mu - 1 = 0 \Leftrightarrow \alpha = \frac{\mu + \sqrt{\mu^2 + 4}}{2}$$

since $\alpha > 0$.

ii. No choice $\alpha = \mu$

b.3- You have already found the M for each of the two cases ($\alpha > \mu$ or $\alpha = \mu$). We use the notation: $p(y)$ is our truncated normal (our target distribution), and $g(y)$ is the shifted exponential that we use as the instrumental density. M is the smallest value of the upper bound of the ratio p/g. So the steps in the algorithm are straightforward:

i. Draw a candidate value X from $g(x)$

ii. Compute the ratio p/Mg evaluated at X

iii. Draw a uniform $U \sim \mathrm{U}(0, 1)$

iv. Set $Y = X$ with probability p/gM

v. Go back to step (i).

Exercise 2

(a) Let $\theta = (\alpha, \beta, \sigma^2)$, $\theta_{-\alpha} = (\beta, \sigma^2)$, and so on.

- Likelihood

$$p(y|\theta) \propto \sigma^{-n} \exp\left(-\frac{1}{2\sigma^2}\sum_i^n (y_i - \alpha - \beta x_i)^2\right)$$

- Priors

$$p(\alpha) \propto \sigma_a^{-1} \exp\left(-\frac{(\alpha - \alpha_0)^2}{2\sigma_a^2}\right)$$

$$p(\beta) \propto \sigma_b^{-1} \exp\left(-\frac{(\beta - \beta_0)^2}{2\sigma_b^2}\right)$$

$$p(\sigma^2) \propto (\sigma^2)^{-(\nu_0/2+1)} \exp\left(\frac{\nu_0 s_0^2}{2\sigma^2}\right)$$

- Joint posterior

$$p(\theta|y) \propto \sigma^{-(\nu_0+2+n)} \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma^2}\left(\sum_i^n (y_i-\alpha-\beta x_i)^2+\nu_0 s_0^2\right)+\frac{(\beta^2-2\beta\beta_0)}{\sigma_b^2} + \frac{(\alpha^2-2\alpha\alpha_0)}{\sigma_a^2}\right]\right\}$$

(b) Conditional posterior distributions

- σ^2

$$\sigma^2|y,\theta_{-\sigma^2} \sim \text{Inv-}\chi^2(\nu_0+n, \sum_i^n (y_i-\alpha-\beta x_i)^2+\nu_0 s_0^2)$$

- β.

$$\beta|y,\theta_{-\beta} \sim \text{N}(\mu_\beta, \sigma_\beta^2)$$

where

$$\mu_\beta = \frac{\sigma_b^2(\sum x_i y_i - n\alpha\bar{x})+\sigma^2\beta_0}{\sigma_b^2\sum x_i^2+\sigma^2} \qquad \sigma_\beta = \frac{\sigma^2\sigma_b^2}{\sigma_b^2\sum x_i^2+\sigma^2}$$

- α.

$$\alpha|y,\theta_{-\alpha} \sim \text{N}(\mu_\alpha, \sigma_\alpha^2)$$

where

$$\mu_\alpha = \frac{\sigma_a^2 n(\bar{y}-\beta\bar{x})+\sigma^2\alpha_0}{n\sigma_a^2+\sigma^2} \qquad \sigma_\alpha = \frac{\sigma^2\sigma_a^2}{n\sigma_a^2+\sigma^2}$$

(c) To obtain a sample of size m using Gibbs sampling do:

1- Choose two of the three parameters, say, β and α. Assign initial values for those parameters, say β_0 and α_0.

2- Set $t=1$.

3- Sample σ_t^2 from a Inv-$\chi^2(\nu_0+n, \sum_i^n (y_i-\alpha_{t-1}-\beta_{t-1}x_i)^2+\nu_0 s_0^2)$ distribution.

4- Sample α_t from a $\text{N}(\mu_\alpha, \sigma_\alpha^2)$ distribution with

$$\mu_\alpha = \frac{\sigma_a^2 n(\bar{y}-\beta_{t-1}\bar{x})+\sigma_t^2\alpha_0}{n\sigma_a^2+\sigma_t^2} \qquad \sigma_\alpha = \frac{\sigma_t^2\sigma_a^2}{n\sigma_a^2+\sigma_t^2}$$

5- Sample β_t from a $\text{N}(\mu_\beta, \sigma_\beta^2)$ distribution with

$$\mu_\beta = \frac{\sigma_b^2(\sum x_i y_i - n\alpha_t\bar{x})+\sigma_t^2\beta_0}{\sigma_b^2\sum x_i^2+\sigma_t^2} \qquad \sigma_\beta = \frac{\sigma_t^2\sigma_b^2}{\sigma_b^2\sum x_i^2+\sigma_t^2}$$

6- Set t = t+1. Go back to step (3) and repeat m times.

(d) WinBugs code

```
model{ for (i in 1:N){ mu[i] <- alpha + beta*x[i]
                       y[i] ~ dnorm(mu[i],tau.y) }
       alpha ~ dnorm(mu.a,tau.a)
       beta  ~ dnorm(mu.b,tau.b)
       tau.y0 ~ dchisq(nu0)
       tau.y <- nu0*s20/tau.y0
       sigma2 <- 1/tau.y    }
```

Below you will find how the data and inital values specification should look like. Note that this part was not required.

```
Data
list(y = c(....),x = c(....), N= ,
     mu.a = , tau.a = , mu.b = , tau.b = , nu0 = , s20 = , )

Inits
list(alpha = , beta = , tau.y0 = )
```