Stat 544 – Spring 2005
Homework assignment 4 answer key

Exercise 1

(a) Let $X$ be the number of simulations needed to produce one observation from the truncated normal. So $X \sim \text{Geometric}(p)$, where $p$ is the acceptance rate, i.e. the probability of obtaining one observation $\geq \alpha$. So $p = 1 - \Phi(\alpha)$. Note that $\text{Range}(X) = \{1, 2, \ldots\}$. Hence, the average number of simulations is $E(X) = (1 - \Phi(\alpha))^{-1}$

(b) Note that $\alpha$ has been replaced by $\mu$ for the sake of clarity

b.1- Upper bound

$$f(x) = \frac{e^{-x^2/2}}{\alpha e^{-\alpha(x-\mu)}} \Rightarrow h(x) = \frac{1}{\alpha} e^{\alpha(x-\mu)} e^{-x^2/2}$$

Thus, $h(x)$ attains its maximum when $x = \alpha$. Hence, the upper bound ignoring truncation is

$$\frac{1}{\alpha} \mu^{\alpha^2/2-\alpha \mu}$$

Now, if we introduce the truncation we only need to consider values of $\alpha$ such that $\alpha \geq \mu$. So, we consider two cases for the bound:

i. $\alpha > \mu$: Upper bound is $\frac{1}{\alpha} \mu^{\alpha^2/2-\alpha \mu} = K(\alpha)$

ii. $\alpha = \mu$: Upper bound is $\frac{1}{\mu} e^{-\mu^2/2} = \frac{\mu}{e^{\mu^2/2}}$

b.2- Now need to find $\alpha$ that minimizes upper bound in each case

i. $\frac{\partial}{\partial \alpha} K(\alpha) = K'(\alpha) = -\frac{1}{\alpha^2} \mu^{\alpha^2/2-\alpha \mu} + \frac{1}{\alpha} \mu^{\alpha^2/2-\alpha \mu} (\alpha - \mu) = \frac{\mu^{\alpha^2/2-\alpha \mu}}{\alpha^2} (\alpha^2 - \alpha \mu - 1)$


\[ K'(\alpha) = 0 \Leftrightarrow \alpha^2 - \alpha \mu - 1 = 0 \Leftrightarrow \alpha = \mu + \sqrt{\mu^2 + 4} \]

since $\alpha > 0$.

ii. No choice $\alpha = \mu$

b.3- You have already found the $M$ for each of the two cases ($\alpha > \mu$ or $\alpha = \mu$). We use the notation: $p(y)$ is our truncated normal (our target distribution), and $q(y)$ is the shifted exponential that we use as the instrumental density. $M$ is the smallest value of the upper bound of the ratio $p/y$. So the steps in the algorithm are straightforward:

i. Draw a candidate value $X$ from $q(x)$

ii. Compute the ratio $p/M$ evaluated at $X$

iii. Draw a uniform $U \sim U(0,1)$

iv. Set $Y = X$ with probability $p/M$

v. Go back to step (i)

Exercise 2

(a) Let $\theta = (\alpha, \beta, \sigma^2)$, $\theta_{\mu} = (\beta, \sigma^2)$, and so on.

- **Likelihood**

  $p(y|\theta) \propto \sigma^{-\nu} \exp \left( -\frac{1}{2\sigma^2} \sum_{i} (y_i - \alpha - \beta x_i)^2 \right)$

- **Priors**

  $p(\alpha) \propto \sigma^{-1} \exp \left( -\frac{(\alpha - \alpha_0)^2}{2\sigma^2} \right)$

  $p(\beta) \propto \sigma^{-1} \exp \left( -\frac{(\beta - \beta_0)^2}{2\sigma^2} \right)$

  $p(\sigma^2) \propto (\sigma^2)^{-\nu/2 - 1} \exp \left( \frac{n\bar{y}^2}{2\sigma^2} \right)$
Joint posterior

\[ p(\theta | y) \propto \sigma^{-\nu_0 - 2 + n} \exp \left\{ -\frac{1}{2} \left[ \frac{1}{\sigma^2} \left( \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2 + n \sigma_0^2 \right) \right] \right\} \]

(b) Conditional posterior distributions

- \( \sigma^2 \)
  \[ \sigma^2_{y, \theta_{-\sigma^2}} \sim \text{Inv-\chi}^2(\nu_0 + n, \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2 + \nu_0 \sigma_0^2) \]

- \( \beta \)
  \[ \beta_{y, \theta_{-\beta}} \sim N(\mu_{\beta}, \sigma_{\beta}^2) \]
  where
  \[ \mu_{\beta} = \frac{\sigma_b^2 \left( \sum x_i y_i - n \alpha \bar{x} \right) + \sigma_b^2 \beta_0}{\sigma_b^2 \sum x_i^2 + \sigma_b^2} \]
  \[ \sigma_{\beta} = \sqrt{\frac{\sigma_b^2 \sigma_b^2}{\sigma_b^2 \sum x_i^2 + \sigma_b^2}} \]

- \( \alpha \)
  \[ \alpha_{y, \theta_{-\alpha}} \sim N(\mu_{\alpha}, \sigma_{\alpha}^2) \]
  where
  \[ \mu_{\alpha} = \frac{\sigma_a^2 \left( \sum x_i y_i - n \beta \bar{x} \right) + \sigma_a^2 \alpha_0}{\sigma_a^2 \sum x_i^2 + \sigma_a^2} \]
  \[ \sigma_{\alpha} = \sqrt{\frac{\sigma_a^2 \sigma_a^2}{\sigma_a^2 \sum x_i^2 + \sigma_a^2}} \]

(c) To obtain a sample of size \( m \) using Gibbs sampling do:

1. Choose two of the three parameters, say, \( \beta \) and \( \alpha \). Assign initial values for those parameters, say \( \beta_0 \) and \( \alpha_0 \).
2. Set \( t = 1 \).
3. Sample \( \sigma_t^2 \) from a Inv-\chi^2(\nu_0 + n, \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2 + \nu_0 \sigma_0^2) \) distribution.
4. Sample \( \alpha_t \) from a \( N(\mu_{\alpha}, \sigma_{\alpha}^2) \) distribution with
  \[ \mu_{\alpha} = \frac{\sigma_a^2 \left( \sum x_i y_i - n \beta \bar{x} \right) + \sigma_a^2 \alpha_0}{\sigma_a^2 \sum x_i^2 + \sigma_a^2} \]
  \[ \sigma_{\alpha} = \sqrt{\frac{\sigma_a^2 \sigma_a^2}{\sigma_a^2 \sum x_i^2 + \sigma_a^2}} \]
5. Sample \( \beta_t \) from a \( N(\mu_{\beta}, \sigma_{\beta}^2) \) distribution with
  \[ \mu_{\beta} = \frac{\sigma_b^2 \left( \sum x_i y_i - n \alpha \bar{x} \right) + \sigma_b^2 \beta_0}{\sigma_b^2 \sum x_i^2 + \sigma_b^2} \]
  \[ \sigma_{\beta} = \sqrt{\frac{\sigma_b^2 \sigma_b^2}{\sigma_b^2 \sum x_i^2 + \sigma_b^2}} \]
6. Set \( t = t+1 \). Go back to step (3) and repeat \( m \) times.

(d) WinBugs code

```winbugs
model{
  for (i in 1:N){
    mu[i] <- alpha + beta*x[i]
    y[i] ~ dnorm(mu[i],tau.y)
  }
  alpha ~ dnorm(mu.a,tau.a)
  beta ~ dnorm(mu.b,tau.b)
  tau.y0 ~ dchisq(nu0)
  tau.y <- nu0*s20/tau.y0
  sigma2 <- 1/tau.y
}
```

Below you will find how the data and initial values specification should look like. Note that this part was not required.

Data

```r
list(y = c(....),x = c(....), N= , mu.a=,tau.a=,mu.b=,tau.b=,nu0=,s20= ,)
```

Inits

```r
list(alpha= , beta= , tau.y0 = )
```