

**Stat 544 – Spring 2005**  
**Homework assignment 4 answer key**

**Exercise 1**

(a) Let  $X$  be the number of simulations needed to produce one observation from the truncated normal. So  $X \sim \text{Geometric}(p)$ , where  $p$  is the acceptance rate, i.e. the probability of obtaining one observation  $\geq a$ . So  $p = 1 - \Phi(a)$ . Note that  $\text{Range}(X) = \{1, 2, \dots\}$ . Hence, the average number of simulations is  $E(X) = [1 - \Phi(a)]^{-1}$

(b) Note that  $a$  has been replaced by  $\mu$  for the sake of clarity

b.1- Upper bound

$$\left. \begin{aligned} f(x) &= e^{-x^2/2} \\ g(x) &= \alpha e^{-\alpha(x-\mu)} \end{aligned} \right\} \Rightarrow h(x) = \frac{1}{\alpha} e^{\alpha(x-\mu)} e^{-x^2/2}$$

$$\frac{\partial}{\partial x} h(x) = \alpha e^{-x^2/2} e^{\alpha^2 - \alpha\mu}$$

Thus,  $h(x)$  attains its maximum when  $x = \alpha$ . Hence, the upper bound ignoring truncation is

$$\frac{1}{\alpha} e^{\alpha^2/2 - \alpha\mu}$$

Now, if we introduce the truncation we only need to consider values of  $\alpha$  such that  $\alpha \geq \mu$ . So, we consider two cases for the bound:

- i.  $\alpha > \mu$ : Upper bound is  $\frac{1}{\alpha} e^{\alpha^2/2 - \alpha\mu} := K(\alpha)$
- ii.  $\alpha = \mu$ : Upper bound is  $\frac{1}{\alpha} e^{-\alpha^2/2} = \frac{1}{\mu} e^{-\mu^2/2}$

b.2- Now need to find  $\alpha$  that minimizes upper bound in each case

i.

$$\frac{\partial}{\partial \alpha} K(\alpha) = K'(\alpha) = -\frac{1}{\alpha^2} e^{\alpha^2/2 - \alpha\mu} + \frac{1}{\alpha} e^{\alpha^2/2 - \alpha\mu} (\alpha - \mu) = \frac{e^{\alpha^2/2 - \alpha\mu}}{\alpha^2} (\alpha^2 - \alpha\mu - 1)$$

$$K'(\alpha) = 0 \Leftrightarrow \alpha^2 - \alpha\mu - 1 = 0 \Leftrightarrow \alpha = \frac{\mu + \sqrt{\mu^2 + 4}}{2}$$

since  $\alpha > 0$ .

ii. No choice  $\alpha = \mu$

b.3- You have already found the M for each of the two cases ( $\alpha > \mu$  or  $\alpha = \mu$ ). We use the notation:  $p(y)$  is our truncated normal (our target distribution), and  $g(y)$  is the shifted exponential that we use as the instrumental density. M is the smallest value of the upper bound of the ratio  $p/g$ . So the steps in the algorithm are straightforward:

- i. Draw a candidate value  $X$  from  $g(x)$
- ii. Compute the ratio  $p/Mg$  evaluated at  $X$
- iii. Draw a uniform  $U \sim U(0, 1)$
- iv. Set  $Y = X$  with probability  $p/gM$
- v. Go back to step (i).

**Exercise 2**

(a) Let  $\theta = (\alpha, \beta, \sigma^2)$ ,  $\theta_{-\alpha} = (\beta, \sigma^2)$ , and so on.

• Likelihood

$$p(y|\theta) \propto \sigma^{-n} \exp\left(-\frac{1}{2\sigma^2} \sum_i^n (y_i - \alpha - \beta x_i)^2\right)$$

• Priors

$$p(\alpha) \propto \sigma_a^{-1} \exp\left(-\frac{(\alpha - \alpha_0)^2}{2\sigma_a^2}\right)$$

$$p(\beta) \propto \sigma_b^{-1} \exp\left(-\frac{(\beta - \beta_0)^2}{2\sigma_b^2}\right)$$

$$p(\sigma^2) \propto (\sigma^2)^{-(\nu_0/2+1)} \exp\left(-\frac{\nu_0 s_0^2}{2\sigma^2}\right)$$

- Joint posterior

$$p(\theta|y) \propto \sigma^{-(\nu_0+2+n)} \exp \left\{ -\frac{1}{2} \left[ \frac{1}{\sigma^2} (\sum_i^n (y_i - \alpha - \beta x_i)^2 + \nu_0 s_0^2) + \frac{(\beta^2 - 2\beta\beta_0)}{\sigma_b^2} + \frac{(\alpha^2 - 2\alpha\alpha_0)}{\sigma_a^2} \right] \right\}$$

(b) Conditional posterior distributions

- $\sigma^2$

$$\sigma^2|y, \theta_{-\sigma^2} \sim \text{Inv-}\chi^2(\nu_0 + n, \sum_i^n (y_i - \alpha - \beta x_i)^2 + \nu_0 s_0^2)$$

- $\beta$ .

$$\beta|y, \theta_{-\beta} \sim N(\mu_\beta, \sigma_\beta^2)$$

where

$$\mu_\beta = \frac{\sigma_b^2 (\sum x_i y_i - n\alpha\bar{x}) + \sigma^2 \beta_0}{\sigma_b^2 \sum x_i^2 + \sigma^2} \quad \sigma_\beta = \frac{\sigma^2 \sigma_b^2}{\sigma_b^2 \sum x_i^2 + \sigma^2}$$

- $\alpha$ .

$$\alpha|y, \theta_{-\alpha} \sim N(\mu_\alpha, \sigma_\alpha^2)$$

where

$$\mu_\alpha = \frac{\sigma_a^2 n(\bar{y} - \beta\bar{x}) + \sigma^2 \alpha_0}{n\sigma_a^2 + \sigma^2} \quad \sigma_\alpha = \frac{\sigma^2 \sigma_a^2}{n\sigma_a^2 + \sigma^2}$$

(c) To obtain a sample of size  $m$  using Gibbs sampling do:

- 1- Choose two of the three parameters, say,  $\beta$  and  $\alpha$ . Assign initial values for those parameters, say  $\beta_0$  and  $\alpha_0$ .
- 2- Set  $t = 1$ .
- 3- Sample  $\sigma_t^2$  from a  $\text{Inv-}\chi^2(\nu_0 + n, \sum_i^n (y_i - \alpha_{t-1} - \beta_{t-1} x_i)^2 + \nu_0 s_0^2)$  distribution.

- 4- Sample  $\alpha_t$  from a  $N(\mu_\alpha, \sigma_\alpha^2)$  distribution with

$$\mu_\alpha = \frac{\sigma_a^2 n(\bar{y} - \beta_{t-1} \bar{x}) + \sigma_t^2 \alpha_0}{n\sigma_a^2 + \sigma_t^2} \quad \sigma_\alpha = \frac{\sigma_t^2 \sigma_a^2}{n\sigma_a^2 + \sigma_t^2}$$

- 5- Sample  $\beta_t$  from a  $N(\mu_\beta, \sigma_\beta^2)$  distribution with

$$\mu_\beta = \frac{\sigma_b^2 (\sum x_i y_i - n\alpha_t \bar{x}) + \sigma_t^2 \beta_0}{\sigma_b^2 \sum x_i^2 + \sigma_t^2} \quad \sigma_\beta = \frac{\sigma_t^2 \sigma_b^2}{\sigma_b^2 \sum x_i^2 + \sigma_t^2}$$

- 6- Set  $t = t+1$ . Go back to step (3) and repeat  $m$  times.

(d) WinBugs code

```
model{ for (i in 1:N){ mu[i] <- alpha + beta*x[i]
                    y[i] ~ dnorm(mu[i],tau.y) }
alpha ~ dnorm(mu.a,tau.a)
beta ~ dnorm(mu.b,tau.b)
tau.y0 ~ dchisq(nu0)
tau.y <- nu0*s20/tau.y0
sigma2 <- 1/tau.y }
```

Below you will find how the data and initial values specification should look like. Note that this part was not required.

Data

```
list(y = c(...), x = c(...), N = ,
      mu.a = , tau.a = , mu.b = , tau.b = , nu0 = , s20 = , )
```

Inits

```
list(alpha = , beta = , tau.y0 = )
```