Problem 1

We wish to obtain draws \( y \) from a truncated (below) normal distribution with mean \( \mu \) and variance \( \sigma^2 \). Suppose that \( \mu = 0 \) and \( \sigma^2 = 1 \), so that

\[
p(y|\mu, \sigma^2) \propto \exp\left(-\frac{y^2}{2}\right), \quad y \geq a
\]

The naive method to sample from this distribution, is simply to draw values from the \( N(0,1) \) and discard all draws that are less than \( a \). This method may be effective if \( a \) is small relative to \( \mu \), but can get costly otherwise.

a - In the example above, what is the average number of simulations from \( N(0,1) \) required for each acceptance?

b - A potential instrumental or proposal distribution in this case is the translated exponential distribution, with pdf

\[
g_\alpha(z) = \alpha \exp\left[-\alpha(z-a)\right], \quad z \geq a
\]

for some value of \( \alpha \).

b.1 - Derive the upper bound on the ratio \( p/g_\alpha(z) \). Hint: There are two upper bounds on the ratio, depending on whether \( \alpha > \alpha \) or not.

b.2 - Find the value of \( \alpha \) that minimizes the upper bound in each case.

b.3 - Given your results in part (b.2), propose an algorithm to sample values from a truncated normal distribution using the rejection sampling method. List all of the steps that are needed to carry out your algorithm.

Problem 2

Consider a simple linear regression model

\[
y_i = \alpha + \beta x_i + \epsilon_i, \quad \epsilon_i \sim N(0, \sigma^2),
\]

and the \( x_i \) are known and fixed. Assume that the errors are independent.

Let the prior distributions for \( \alpha \) and \( \beta \) be \( N(\alpha_0, \sigma^2) \) and \( N(\beta_0, \sigma^2) \), respectively, with parameters fixed and known. For \( \sigma^2 \), let the prior distribution be an inverted scaled \( \chi^2 \) distribution with known parameters.

We are interested in estimating the posterior distributions of \( \alpha \), \( \beta \), and \( \sigma^2 \).