Problem 1

```r
sampleBeta22 <- function(n) {
    values <- NULL # set up space for the values to be returned
    for (i in 1:n) { # loop n times (for a sample of size n)
        # put 2 U[0,1] r.v.s into the vector U
        U <- runif(2)
        # resample these uniform r.v.s until
        while ( U[2] > 4 * U[1] * (1 - U[1]) ) { U <- runif(2) }
        values[i] <- U[1] # store U[1] as values[i]
    } # end loop
    return(values) # return our sample
}
```

Problem 2

a-

\[
M = \frac{\alpha^\alpha \exp(1 - \alpha)}{\Gamma(\alpha)}
\]

b- For \( \alpha = 2 \)

![Target density with envelope](image-url)
c- The following R-code can be used to obtain the sample. Note that we have chosen $\alpha = 2$

```r
#rejection method for Gamma(alpha,1)
Gamma.alpha.1 <- function(m,alpha){
  x<- NULL
  h <- 1/alpha*exp(-x/alpha)
  M <- alpha^alpha*exp(1-alpha)/gamma(alpha)
  for (i in 1:m){repeat { u <- runif(1)
    z <- rexp(1,1/alpha)
    if(u<= dgamma(z,2)/(M/alpha*exp(-z/alpha))){x[i]<-z
      break }
  } #ends repeat
} #ends for
x<-Gamma.alpha.1(1000,2)
hist(x, main="Sample of Gamma(2,1) via rejection method",
     xlab="x",nclass=30)
```

d- alpha <- seq(2,50,length=100)
trial_ratio <- 1/(alpha^alpha*exp(1-alpha)/gamma(alpha))

![Acceptance probability](image)

Problem 3

A copy of the R code can be found in the class web page.
Problem 4

a - A copy of the R code can be found in the class web page.

b - WinBUGS code

```r
model { for (i in 1:N) { y[i] ~ dnorm(mu, tau) }

sg2 <- (v.p*sg2.p)/X
tau <- 1/sg2
X ~ dchisqr(v.p)
}
list( y = c( 375, ..., 437), N = 100, mu = 404.59, v.p = 10, sg2.p = 40 )
```

c - Results

<table>
<thead>
<tr>
<th>Source</th>
<th>95% credible set</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>lower bound</td>
</tr>
<tr>
<td>Metropolis-Hasting</td>
<td>18.80</td>
</tr>
<tr>
<td>WinBUGS</td>
<td>30.54</td>
</tr>
</tbody>
</table>

The difference is mostly due to the fact that 1000 iterations are not enough for the Metropolis-Hasting algorithm to converge. Note that WinBUGS is not affected by convergence problems since WinBUGS, in this special case, is drawing directly from the posterior distribution of $\sigma^2$.

Problem 5

Let $\theta = (\theta_1, \ldots, \theta_n)$ and $y = (y_1, \ldots, y_n)$.

a - $p(\theta, \phi|y) \propto p(y|\theta, \phi)p(\theta|\phi)p(\phi)$

$$p(\theta, \phi|y) = \phi^n \exp\left(-\left(1 + \phi\right)\sum_{i=1}^{n} \theta_i \right) \prod_{i=1}^{n} \theta_i \theta_i!$$

b - $p(\theta, z|y) \propto p(y|\theta, z)p(\theta|z)p(z)$

- $p(z)$

$$z = g(\phi) = \frac{1}{1 + \phi} \Rightarrow \phi = g^{-1}(z) = \frac{1 - z}{z} \Rightarrow \frac{\partial g^{-1}(z)}{\partial z} = -\frac{1}{z^2}$$

$$p(\phi) \propto 1 \Rightarrow p(z) \propto \frac{1}{z^2} \text{ for } z \in (0, 1]$$
\[ p(\theta|z) = \frac{(1-z)^n}{z^n} \exp \left\{ -\frac{1}{z} \sum_{i=1}^{n} \theta_i \right\} \]

\[ p(y|\theta) = \prod_{i=1}^{n} \theta_i^{y_i}e^{-\theta_i/y_i} = \exp(-\sum_{i=1}^{n} \theta_i) \prod_{i=1}^{n} \theta_i^{y_i} \]

Thus,

\[ p(\theta, z|y) \propto \left( \prod_{i=1}^{n} \theta_i^{y_i} \right) \frac{(1-z)^n}{z^n} \exp \left\{ -\frac{1}{z} \sum_{i=1}^{n} \theta_i \right\} \]

Now,

\[ p(z|y) = \int_{\mathbb{R}^n} p(\theta, z|y) d\theta \propto \int_{\mathbb{R}^n} \left( \prod_{i=1}^{n} \theta_i^{y_i} \right) \frac{(1-z)^n}{z^n} \exp \left\{ -\frac{1}{z} \sum_{i=1}^{n} \theta_i \right\} d\theta \]

and since

\[ \prod_{i=1}^{n} \int_{0}^{+\infty} \theta_i^{y_i} \exp \left\{ -\frac{1}{z} \theta_i \right\} d\theta_i = \prod_{i=1}^{n} \Gamma(y_i + 1) z^{y_i + 1} = z^{\sum_{i=1}^{n} y_i + n} \]

we have

\[ p(z|y) \propto \frac{(1-z)^n z^{\sum_{i=1}^{n} y_i} z^{\sum_{i=1}^{n} n} = (1-z)^n z^{\sum_{i=1}^{n} y_i + 1}}{z^n} \]

Hence, provided that \( n\bar{y} > 1 \), \( (1-z)^n z^{\sum_{i=1}^{n} y_i + 1} \) is the kernel of a Beta distribution. Thus, \( z|y \sim \text{Beta}(n\bar{y} - 1, n + 1) \) if \( n\bar{y} > 1 \).

c. Note that from what we have found on part (a)

\[ E(z|y) = \frac{n\bar{y} - 1}{n\bar{y} + n} \]

Further,

\[ p(\theta_i|y, z) \propto \theta_i^{y_i}e^{-\theta_i/z} \Rightarrow \theta_i|y_i, z \sim \text{Gamma}(y_i + 1, z^{-1}) \Rightarrow E(\theta_i|y_i, z) = (y_i + 1)z \]

So,

\[ E(\theta_i|y) = E[E(\theta_i|y, z)] = E[(y_i + 1)z|y] = (y_i + 1)E(z|y) = (y_i + 1) \frac{n\bar{y} - 1}{n\bar{y} + n} \]
d - Note that \( n\bar{y} < 1 \Rightarrow \sum_{i=1}^{n} y_i < 1 \) and since \( y_i \in \{0, 1, 2, \ldots\} \forall i \)

\[
\sum_{i=1}^{n} y_i < 1 \iff \sum_{i=1}^{n} y_i = 0 \iff y_1 = \ldots = y_n = 0
\]

Let \( y_0 = (0, \ldots, 0) \), then the posterior when \( y = y_0 \) becomes

\[
p(\theta, z|y) \propto \frac{(1 - z)^n}{z^{n+2}} \exp \left\{ -\frac{\sum_{i=1}^{n} \theta_i}{z} \right\}
\]

Whence,

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^n} p(\theta, z|y_0) d\theta \, dz \propto \int_{0}^{1} \int_{\mathbb{R}^n} \frac{(1 - z)^n}{z^{n+2}} \exp \left\{ -\frac{1}{z} \sum_{i=1}^{n} \theta_i \right\} d\theta \, dz =
\]

\[
= \int_{0}^{1} \frac{(1 - z)^n}{z^{n+2}} \left[ \prod_{i=1}^{n} \int_{0}^{+\infty} e^{-\theta_i z} d\theta_i \right] dz = \int_{0}^{1} \frac{(1 - z)^n}{z^{n+2}} z^n dz = \int_{0}^{1} \frac{(1 - z)^n}{z^2} dz
\]

The last integral is divergent. Thus, the posterior distribution of \( \theta, z|y \) at \( y = y_0 \) is improper. Alternatively, you could have used the result from part (b) to show that

\( z|y = y_0 \sim \text{Beta}(-1, n + 1) \)

which is an improper distribution.

When \( n\bar{y} = 1 \Rightarrow \sum_{i=1}^{n} y_i = 1 \) which implies that \( y_i = 1 \) for some \( i \) and \( y_j = 0 \forall j \neq i \).

So, the joint posterior is also improper.