Extended hint for problem 3, part c of hw 2

You have to “fill in the blanks”, i.e. you have to justify each step that has been shown here.

The hint suggest you to work with $X_1 = Y_1 - Y_2$ and $X_2 = Y_1 + Y_2$ instead of with $Y_1$ and $Y_2$.

You know \[
\begin{pmatrix}
Y_1 \\
Y_2
\end{pmatrix}
\sim
N\left(\begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix},
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}\right)
\]

Thus, with $\mu_1 := \theta_1 - \theta_2$ and $\mu_2 := \theta_1 + \theta_2$ we have \[
\begin{pmatrix}
X_1 \\
X_2
\end{pmatrix}
\sim
N\left(\begin{pmatrix}
\mu_1 \\
\mu_2
\end{pmatrix},
\begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}\right)
\] (1)

Further, note the following:

a - $\theta_1 = (\mu_1 + \mu_2)/2$

b - $X_1$ and $X_2$ uniquely determine $Y_1$ and $Y_2$. e.g. $Y_1 = (X_1 + X_2)/2$

c - $\theta_1 > \theta_2 \Rightarrow 0 < \mu_1 < \infty$

d - $-\infty < \mu_2 < \infty$

e - The restriction on $\mu_1$ is independent of $\mu_2$

f - $p(\theta_1, \theta_2) \propto 1 \Rightarrow p(\mu_1, \mu_2) \propto 1$

Thus, part c asks us to calculate:

$$E(\theta_1 | y_1, y_2) = E\left(\frac{\mu_1 + \mu_2}{2} | x_1, x_2\right) = \frac{1}{2}(E(\mu_1 | x_1) + E(\mu_2 | x_2))$$

Now, from (1) we know that $\mu_2 | x_2 \sim N(x_2, 2)$. Thus, $E(\mu_2 | x_2) = x_2$. So, to solve the exercise you only need to find $E(\mu_1 | x_1)$. This involves a little more work, first we have to note that

$$p(\mu_1 | x_1) = \frac{\exp(-((\mu_1 - x_1)^2/4))}{2\sqrt{\pi}\Phi(x_1/\sqrt{2})}I[\mu_1 > 0]$$

and

$$E(\mu_1 | x_1) = E((\mu_1 - x_1 + x_1) | x_1) = E(\mu_1 - x_1 | x_1) + E(x_1 | x_1)$$

\(^1\)the sign := stands for “is by definition equal to”
Also of help will be (you do not have to justify any of this, use them as facts)

a- the following mathematical relationship

\[
\frac{\partial}{\partial x_1} \exp \left\{ -\frac{1}{2} \left( \mu_1 - x_1 \right)^2 \right\} = \frac{\mu_1 - x_1}{2} \exp \left\{ -\frac{1}{2} \left( \mu_1 - x_1 \right)^2 \right\}
\]

b- to note that all the conditions needed to be possible to interchange the integral and derivative signs hold, i.e.

\[
\int \frac{\partial}{\partial y} f(x, y) \, dx = \frac{\partial}{\partial y} \int f(x, y) \, dx
\]