Extended hint for problem 3, part c of hw 2

You have to "fill in the blanks", i.e. you have to justify each step that has been shown here.

The hint suggest you to work with $X_1 = Y_1 - Y_2$ and $X_2 = Y_1 + Y_2$ instead of with Y_1 and Y_2.

You know

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

Thus, with[1] $\mu_1 := \theta_1 - \theta_2$ and $\mu_2 := \theta_1 + \theta_2$ we have

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) \tag{1}$$

Further, note the following:

a- $\theta_1 = (\mu_1 + \mu_2)/2$

b- X_1 and X_2 uniquely determine Y_1 and Y_2. e.g. $Y_1 = (X_1 + X_2)/2$

c- $\theta_1 > \theta_2 \Rightarrow 0 < \mu_1 < \infty$

d- $-\infty < \mu_2 < \infty$

e- The restriction on μ_1 is independent of μ_2

f- $p(\theta_1, \theta_2) \propto 1 \quad \Rightarrow \quad p(\mu_1, \mu_2) \propto 1$

Thus, part c asks us to calculate:

$$\mathrm{E}(\theta_1|y_1, y_2) = \mathrm{E}\left(\frac{\mu_1 + \mu_2}{2}|x_1, x_2\right) = \frac{1}{2}(\mathrm{E}(\mu_1|x_1) + \mathrm{E}(\mu_2|x_2))$$

Now, from (1) we know that $\mu_2|x_2 \sim \mathrm{N}(x_2,2)$. Thus, $\mathrm{E}(\mu_2|x_2) = x_2$. So, to solve the exercise you only need to find $\mathrm{E}(\mu_1|x_1)$. This involves a little more work, first we have to note that

$$p(\mu_1|x_1) = \frac{\exp(-(\mu_1 - x_1)^2/4)}{2\sqrt{\pi}\Phi(x_1/\sqrt{2})} I[\mu_1 > 0]$$

and

$$\mathrm{E}(\mu_1|x_1) = \mathrm{E}((\mu_1 - x_1 + x_1)|x_1) = \mathrm{E}(\mu_1 - x_1|x_1) + \mathrm{E}(x_1|x_1)$$

[1] the sign := stands for "is by definition equal to"

Also of help will be (you do not have to justify any of this, use them as facts)

a- the following mathematical relationship

$$\frac{\partial}{\partial x_1} \exp\left\{-\frac{1}{2}\frac{(\mu_1 - x_1)^2}{2}\right\} = \frac{(\mu_1 - x_1)}{2} \exp\left\{-\frac{1}{2}\frac{(\mu_1 - x_1)^2}{2}\right\}$$

b- to note that all the conditions needed to be possible to interchange the intergral and derivative signs hold, i.e.

$$\int \frac{\partial}{\partial y} f(x, y)\, dx = \frac{\partial}{\partial y} \int f(x, y)\, dx$$