Chapter 1
Ordinary Differential Equation

1.1 String Subject to Transverse Loading

Consider a string under tension T that is stretched along the x-axis with its end points fixed at $x = 0$ and $x = l$. A transverse distributed load $f(x)$ is applied along the length of the string. The load $f(x)$ is measured with the unit of force/length. Considering a segment of the string,

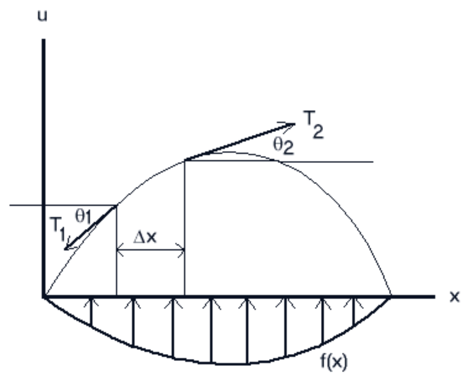

Figure 1

the horizontal and vertical equilibrium equations can be written as

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 = T \tag{1.1}$$

$$T_2 \sin\theta_2 + f(x)\Delta x = T_1 \sin\theta_1 \tag{1.2}$$

These two equations can be combined to yield

$$T(\tan\theta_2 - \tan\theta_1) = -f(x)\Delta x \tag{1.3}$$

Denoting the deflection of the string by $u(x)$, one can write du/dx for $\tan\theta$ and re-write Eq.(1.3) as

$$T\left(\frac{\frac{du}{dx}\big|_2 - \frac{du}{dx}\big|_1}{\Delta x} \right) = -f(x) \tag{1.4}$$

In the limit $\Delta x \to 0$, Eq.(1.4) becomes

$$T\frac{d^2u}{dx^2} = -f(x) \tag{1.5}$$

The deflection $u(x)$ must satisfy the fixed-end conditions $u(0) = 0$ and $u(l) = 0$.
Consider the load

$$f(x) = \begin{pmatrix} 0;\ 0 \le x < x_0 - \frac{\varepsilon}{2} \\ p;\ \ |\ x - x_0\ | \le \frac{\varepsilon}{2} \\ 0;\ x_0 + \frac{\varepsilon}{2} < x \le l \end{pmatrix} \tag{1.6}$$

which is shown in Fig.2.

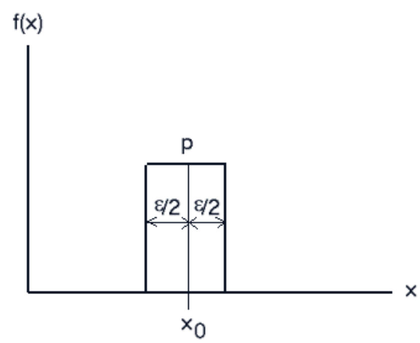

Figure 2

Integrating Eq.(1.5) and inserting the boundary conditions $u(0) = 0$, $u(l) = 0$, we get

$$u(x) = \begin{pmatrix} Ax;\ 0 \le x < x_0 - \frac{\varepsilon}{2} \\ C + Dx - \frac{px^2}{2};\ \ |\ x - x_0\ | \le \frac{\varepsilon}{2} \\ B(l - x);\ x_0 + \frac{\varepsilon}{2} < x \le l \end{pmatrix} \tag{1.7}$$

For this discontinuous $f(x)$, $\frac{d^2u}{dx^2}$ is also discontinuous. However, $\frac{du}{dx}$ and u are continuous. Enforcing these continuity conditions at $x = x_0 - \frac{\varepsilon}{2}$ and $x = x_0 + \frac{\varepsilon}{2}$, we can calculate the constants $A, B, C,$ and D. Thus,

$$u(x) = \frac{p\varepsilon}{T}\begin{pmatrix} \frac{(l-x_0)x}{l};\ 0 \le x < x_0 - \frac{\varepsilon}{2} \\ -\frac{x^2}{2\varepsilon} + x\left(\frac{x_0}{\varepsilon} - \frac{x_0}{l} + \frac{1}{2}\right) - \frac{1}{2\varepsilon}\left(x_0 - \frac{\varepsilon}{2}\right)^2;\ \ |\ x - x_0\ | \le \frac{\varepsilon}{2} \\ \frac{(l-x)x_0}{l};\ x_0 + \frac{\varepsilon}{2} < x \le l \end{pmatrix} \tag{1.8}$$

The solution $u(x)$ is shown in Fig.3.

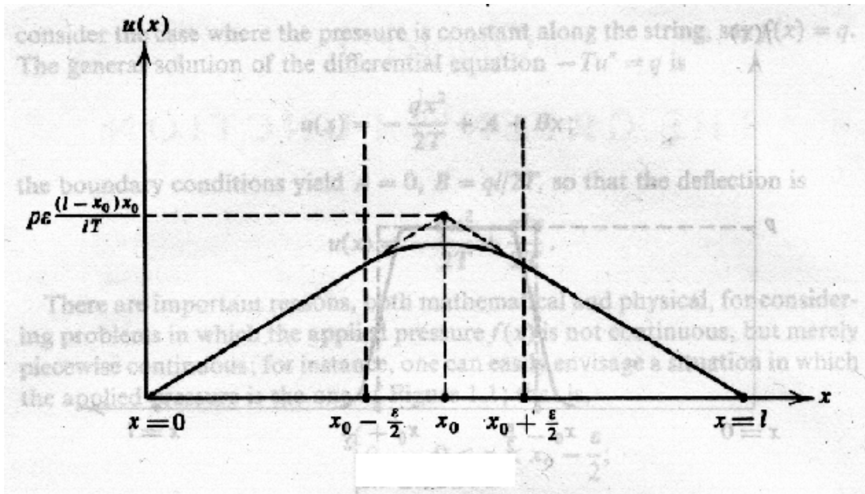

Figure 3

1.2 Principle of Superposition

When a general transverse loading $f(x)$ is given, that transverse loading can be approximated by a superposition of several block loadings as shown in Fig.4.

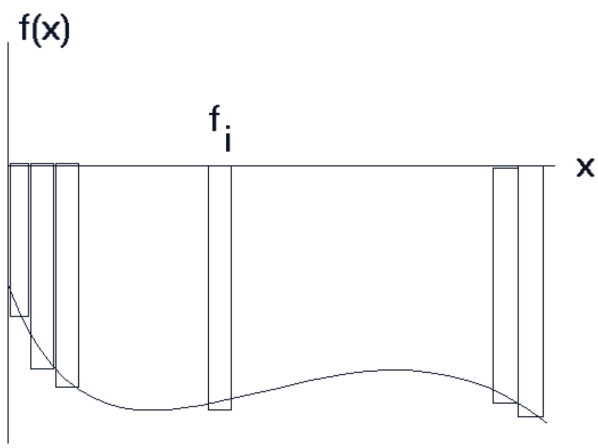

Figure 4

Each of these blocks is of width ε and the deflection for this combined loading can be written as

$$u(x) = \sum u_i(x \mid x_i) \tag{1.9}$$

where $u_i(x \mid x_i)$ is the deflection due to a block loading centered at x_i. When the height of this block is f_i, one can write $u_i(x \mid x_i)$, by utilizing Eq.(1.8), as

$$u_i(x \mid x_i) = \frac{f_i\varepsilon}{T}\left(\begin{array}{l} \frac{(l-x_i)x}{l};\ 0 \le x < x_i - \frac{\varepsilon}{2} \\ -\frac{x^2}{2\varepsilon} + x\left(\frac{x_i}{\varepsilon} - \frac{x_i}{l} + \frac{1}{2}\right) - \frac{1}{2\varepsilon}\left(x_i - \frac{\varepsilon}{2}\right)^2;\quad \mid x - x_i \mid \le \frac{\varepsilon}{2} \\ \frac{(l-x)x_i}{l};\ x_i + \frac{\varepsilon}{2} < x \le l \end{array}\right) \tag{1.10}$$

One can improve the representation of Eq.(1.9) by increasing the number of block loads that are used to represent $f(x)$. Continuously increasing the number of block loads will shrink the width of each block. Use of infinitely many block loads to represent $f(x)$ is equivalent to taking the limit $\varepsilon \to 0$. As $\varepsilon \to 0$, the value $f_i \to f(x_i)$ at such a rate that

$$\lim_{\varepsilon \to 0} f_i\varepsilon = f(x_i)dx_i \tag{1.11}$$

Then the function $u_i(x \mid x_i)$ of Eq.(1.10) becomes

$$u_i(x \mid x_i) = \frac{f(x_i)}{T}\left(\begin{array}{l} \frac{(l-x_i)x}{l};\ 0 \le x < x_i \\ \frac{(l-x)x_i}{l};\ x_i < x \le l \end{array}\right)dx_i \tag{1.12}$$

One can write this expression as

$$u_i(x \mid x_i) = f(x_i)G(x \mid x_i)dx_i$$

where

$$G(x \mid x_i) = \frac{1}{T}\left(\begin{array}{l} \frac{(l-x_i)x}{l};\ 0 \le x < x_i \\ \frac{(l-x)x_i}{l};\ x_i < x \le l \end{array}\right) \tag{1.13}$$

Furthermore, in this limit of $\varepsilon \to 0$, the summation in Eq.(1.9), that includes infinite number of terms, becomes an integration

$$u(x) = \int_0^l f(x_i)G(x \mid x_i)dx_i \tag{1.14}$$

The integral in Eq.(1.14) has to be partitioned into two parts as $G(x|x_i)$ possesses two

different forms for the two ranges $0 \le x < x_i$ and $x_i < x \le l$. Hence

$$u(x) = \int_0^x f(x_i)G(x|x_i)dx_i + \int_x^l f(x_i)G(x|x_i)dx_i$$

In the first integral, $x_i < x$, and in the second integral, $x \le x_i$. Substituting $G(x|x_i)$ from Eq.(1.13), we get

$$u(x) = \int_0^x f(x_i)\frac{(l-x)x_i}{lT}dx_i + \int_x^l f(x_i)\frac{(l-x_i)x}{lT}dx_i$$

Exercise 1.0
Find the solution of

$$\frac{d^2u}{dx^2} = 3x^2$$

where the boundary conditions are $u = 0$ at $x = 0$ and $x = 1$.

1.3 Green’s Function and Delta Function

The function $G(x \mid x_i)$ is known as the Green’s function for this homogeneous problem. Now, let us examine a few properties of the Green’s function. It is easily seen from Eq.(1.13) that the Green’s function satisfies the boundary conditions $G(0 \mid x_i) = 0$ and $G(l \mid x_i) = 0$. A plot of $G(x \mid x_i)$ is shown below.

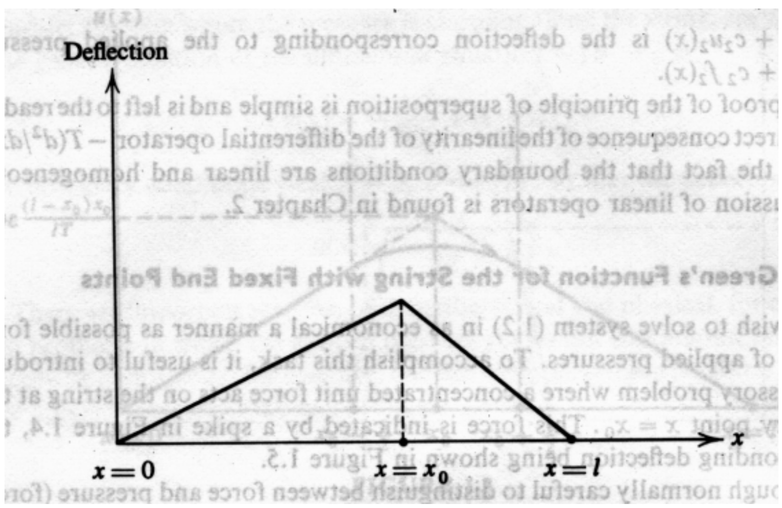

Figure 5

The function $G(x \mid x_i)$ can be interpreted as the deflection of a stretched string due to a point load of unit magnitude (1 Newton or 1 pound). As the point load of unit magnitude is applied over a zero width, the force per unit length is infinite. In addition, one can see from Eq.(1.13)

$$T\frac{d}{dx}G(x \mid x_i) = \left(\begin{array}{l} 1 - \frac{x_i}{l};\ 0 \le x < x_i \\ -\frac{x_i}{l};\ x_i < x \le l \end{array} \right) \tag{1.15}$$

By utilizing Eq.(1.15), it can be shown that

$$T\frac{d}{dx}G(x \mid x_i)\ |_{x=x_i^-} - T\frac{d}{dx}G(x \mid x_i)\ |_{x=x_i^+} = 1 \tag{1.16}$$

where x_i^- and x_i^+ are two points just to the left and just to the right of x_i, respectively. The Eq.(1.16) states that $T\frac{d}{dx}G(x \mid x_i)$ makes a unit jump across the point where the point load is applied. The step function $T\frac{d}{dx}G(x \mid x_i)$ is shown in Fig.6.

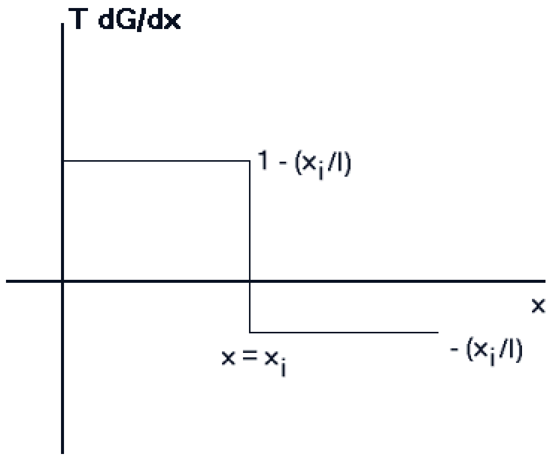

Figure 6

Differentiating both sides of Eq.(1.15), we obtain

$$T\frac{d^2}{dx^2}G(x \mid x_i) = \left(\begin{array}{c} 0; \; x \neq x_i \\ -\infty; \; x = x_i \end{array} \right) \tag{1.17}$$

By using a special symbol for the right-hand-side of Eq.(1.17), we write

$$T\frac{d^2}{dx^2}G(x \mid x_i) = -\delta(x - x_i) \tag{1.18}$$

The function $\delta(x - x_i)$ is known as the Dirac delta function. The delta function represents a point force of magnitude one. This unit point force causes a unit jump in the slope of the Green's function $G(x \mid x_i)$.

To summarize, $u(x)$ is the deflection of the stretched string under the loading $f(x)$, and $G(x \mid x_i)$ is the deflection of the string under a point load of unit magnitude. The Green's function, $G(x \mid x_i)$, satisfies the same boundary conditions as $u(x)$, i.e., zero deflection at the two fixed ends of the string. Furthermore, the solution $u(x)$ can be constructed from $G(x \mid x_i)$ through an integration, as shown in Eq.(1.14).

1.4 Properties of Delta Function

As the delta function represents an unit point force, an integration of the delta function across the point of application of the point load yields unity. However, an integration over an interval that does not include $x = x_i$, the integration yields zero. Referring to Fig.6, one can see

$$\int_a^b \delta(x - x_i)dx = 1; \qquad x_i \; in \; range \; (a, b) \tag{1.19}$$

and

$$\int_b^c \delta(x - x_i)dx = 0; \qquad x_i \textit{ not in range } (b,c) \tag{1.20}$$

The delta function has an important property called the sifting property which is derived as follows.

$$\int_a^b \phi(x)\delta(x-x_i)dx = \int_a^{x_i^-} \phi(x)\delta(x-x_i)dx + \int_{x_i^-}^{x_i^+} \phi(x)\delta(x-x_i)dx + \int_{x_i^+}^{b} \phi(x)\delta(x-x_i)dx$$

$$= \phi(x_i)\int_{x_i^-}^{x_i^+} \delta(x-x_i)dx = \phi(x_i) \tag{1.21}$$

The delta function can be simulated as the limit of a sequence of some analytic functions. Here we mention only one of those - $s_k(x)$.

$$s_k(x) = \frac{1}{\pi}\frac{k}{1+k^2x^2} \tag{1.22}$$

Integrating $s_k(x)$, we obtain

$$r_k(x) = \int_{-\infty}^{x} s_k(u)du = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(kx); \; \textit{for all } k \tag{1.23}$$

Furthermore, it is easy to see that

$$\lim_{k\to\infty} s_k(x) = \begin{pmatrix} 0; \; x \neq 0 \\ \infty; \; x = 0 \end{pmatrix} \tag{1.24}$$

$$\lim_{k\to\infty} r_k(x) = \begin{pmatrix} 0; \; x < 0 \\ \frac{1}{2}; \; x = 0 \\ 1; \; x > 0 \end{pmatrix} \tag{1.25}$$

The Eqs.(1.24) and (1.25) together with Fig.7 where $s_k(x)$ is plotted for various values of k show that $s_k(x)$ has all the characteristics of the delta function as $k \to \infty$.

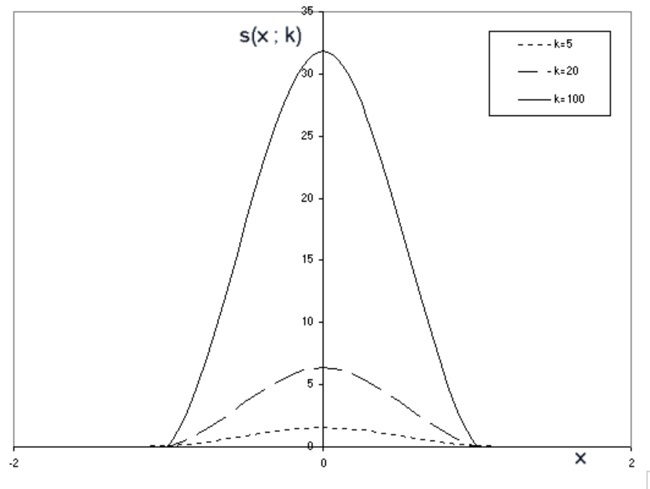

Figure 7

In Fig.8, the function $r_k(x)$ is shown. This function approaches the step-function of Fig.6 as $k \to \infty$.

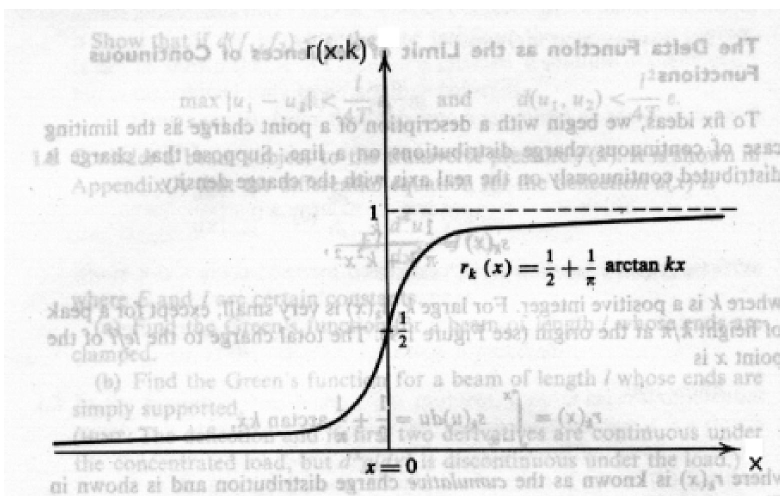

Figure 8

1.5 Mixed Boundary Value Problems

We showed the use of Green's function in constructing the solution of the problem of a loaded string with fixed ends. Now, we extend the method to the loaded string problem with more general boundary conditions. We begin with our problem

$$T\frac{d^2u}{dx^2} = -f(x) \tag{1.26}$$

and the accessory problem

$$T\frac{d^2}{dx^2}G(x \mid x_i) = -\delta(x - x_i) \tag{1.27}$$

At this stage, we will not be concerned with the specified boundary conditions for these problems. We multiply Eq.(1.26) by $G(x \mid x_i)$ and multiply Eq.(1.27) and take the difference to obtain

$$T\left[u(x)\frac{d^2}{dx^2}G(x \mid x_i) - G(x \mid x_i)\frac{d^2u}{dx^2}\right] = -u(x)\delta(x - x_i) + f(x)G(x \mid x_i)$$

Integrating this equation over the length of the cable to get

$$T\int_0^l\left[u(x)\frac{d^2}{dx^2}G(x \mid x_i) - G(x \mid x_i)\frac{d^2u}{dx^2}\right]dx = \tag{1.28}$$
$$-\int_0^l u(x)\delta(x - x_i)dx + \int_0^l f(x)G(x \mid x_i)dx$$

Now, for brevity, we introduce the following notations

$$F = \frac{dG}{dx} \; and \; q = \frac{du}{dx}$$

Then, we integrate the left-hand-side of Eq.(1.28) by parts and employ the sifting property, Eq.(1.21), to the first term on the right-hand-side to obtain

$$T\,[u(l)F(l \mid x_i) - u(0)F(0 \mid x_i) - G(l \mid x_i)q(l) + G(0 \mid x_i)q(0)]$$

$$= -u(x_i) + \int_0^l f(x)G(x \mid x_i)dx \tag{1.29}$$

We now demonstrate the use of Eq.(1.29) for the three, following problems.

Example 1.1*:*
Solve:

$$T\frac{d^2u}{dx^2} = -f(x)\,; \; u(0) = a; \; u(l) = b$$

Accessory problem:

$$T\frac{d^2}{dx^2}G(x \mid x_i) = -\delta(x - x_i)$$

For this problem, $q(0)$ and $q(l)$ are unknown. Thus, we choose the boundary conditions for the accessory problem in such a way that $q(0)$ and $q(l)$ disappear from Eq.(1.29). These boundary conditions are

$$G(l \mid x_i) = G(0 \mid x_i) = 0$$

We have already solved this accessory problem and the solution, from Eq.(1.13), is

$$G(x \mid x_i) = \frac{1}{T}\begin{pmatrix} \frac{(l-x_i)x}{l};\ 0 \le x < x_i \\ \frac{(l-x)x_i}{l};\ x_i < x \le l \end{pmatrix}$$

Thus,

$$F(x \mid x_i) = \frac{d}{dx}G(x \mid x_i) = \frac{1}{T}\begin{pmatrix} \frac{(l-x_i)}{l};\ 0 \le x < x_i \\ -\frac{x_i}{l};\ x_i < x \le l \end{pmatrix}$$

and

$$F(l \mid x_i) = -\frac{x_i}{Tl}\ \ and\ F(0 \mid x_i) = \frac{(l-x_i)}{Tl} \tag{1.30}$$

By using Eq.(1.29), we obtain the solution of our problem as

$$u(x_i) = \frac{a(l-x_i)}{l} + \frac{bx_i}{l} + \frac{1}{Tl}\int_0^{x_i} f(x)(l-x_i)x dx + \frac{1}{Tl}\int_{x_i}^{l} f(x)(l-x)x_i dx \tag{1.31}$$

Example 1.2*:*
Solve:

$$T\frac{d^2u}{dx^2} = -f(x)\ ;\ u(0) = a;\ q(l) = b$$

Accessory problem:

$$T\frac{d^2}{dx^2}G(x \mid x_i) = -\delta(x-x_i)$$

For this problem, $q(0)$ and $u(l)$ are unknown. Thus, we choose the boundary conditions for the accessory problem in such a way that $q(0)$ and $u(l)$ disappear from Eq.(1.29). These boundary conditions are

$$F(l \mid x_i) = G(0 \mid x_i) = 0$$

Integrating the differential equation for the accessory problem, we get

$$TG(x \mid x_i) = \begin{bmatrix} Ax+B;\ x < x_i \\ Cx+D;\ x > x_i \end{bmatrix}$$

Enforcing the condition $G(0 \mid x_i) = 0,$ we get $B = 0$ and enforcing the condition $F(l \mid x_i) = 0$, we get $C = 0$. Thus

$$TG(x \mid x_i) = \begin{bmatrix} Ax;\ x < x_i \\ D;\ x > x_i \end{bmatrix}$$

The jump condition, Eq.(1.16), on the slope of $G(x \mid x_i)$ yields $A = 1$. The condition of continuity of $G(x \mid x_i)$ at $x = x_i$ yields $D = x_i$. Thus,

$$TG(x \mid x_i) = \begin{bmatrix} x;\ x < x_i \\ x_i;\ x > x_i \end{bmatrix} \tag{1.32}$$

From Eq.(1.32), we obtain $TG(l \mid x_i) = x_i$ and $TF(0 \mid x_i) = 1$. We can now utilize Eq.(1.29) to write the solution of our problem as

$$u(x_i) = a + bx_i + \frac{1}{T}\int_0^{x_i} xf(x)dx + \frac{1}{T}\int_{x_i}^{l} x_i f(x)dx$$

Exercise 1.3: For the problem

$$T\frac{d^2u}{dx^2} = -f(x)\ ;\ q(0) = a;\ u(l) = b$$

solve the accessory problem

$$T\frac{d^2}{dx^2}G(x \mid x_i) = -\delta(x - x_i);\ F(0 \mid x_i) = G(l \mid x_i) = 0$$

and write the solution $u(x_i)$.

1.6 Fundamental Solution

In the previous section, we discovered a method to solve mixed boundary value problems for the differential equation

$$T\frac{d^2u}{dx^2} = -f(x)$$

by utilizing the Green's function. One major drawback of this method is that a new Green's function need to be obtained for each type of problem. For complex differential operators, this determination of the Green's function can be laborious. Thus, we now develop a modified scheme where a new Green's function need not be determined when the character of the boundary condition changes. The basis of this method is Eq.(1.29) which we rewrite as follows:

$$u(x_i) = T\left[F(0 \mid x_i)u(0) - F(l \mid x_i)u(l) - G(0 \mid x_i)q(0) + G(l \mid x_i)q(l)\right]$$
$$+ \int_0^l f(x)G(x \mid x_i)dx \tag{1.33}$$

We will call Eq.(1.33) the Boundary Integral Equation (BIE).

Now we write the BIE for two locations by putting x_i very close to the two end points.

These locations are just to the right of the left end point at 0^+ and just to the left of the right end point at l^-.

$$u(0^+) = T\,[F(0 \mid 0^+)u(0) - F(l \mid 0^+)u(l) - G(0 \mid 0^+)q(0) + G(l \mid 0^+)q(l)]$$

$$+ \int_0^l f(x)G(x \mid 0^+)dx$$

and

$$u(l^-) = T\,[F(0 \mid l^-)u(0) - F(l \mid l^-)u(l) - G(0 \mid l^-)q(0) + G(l \mid l^-)q(l)]$$

$$+ \int_0^l f(x)G(x \mid l^-)dx$$

In matrix notation,

$$\begin{pmatrix} u(0^+) \\ u(l^-) \end{pmatrix} = T\begin{bmatrix} -G(0 \mid 0^+) & G(l \mid 0^+) \\ -G(0 \mid l^-) & G(l \mid l^-) \end{bmatrix}\begin{pmatrix} q(0) \\ q(l) \end{pmatrix}$$

$$+ T\begin{bmatrix} F(0 \mid 0^+) & -F(l \mid 0^+) \\ F(0 \mid l^-) & -F(l \mid l^-) \end{bmatrix}\begin{pmatrix} u(0) \\ u(l) \end{pmatrix} + \begin{pmatrix} \int_0^l f(x)G(x \mid 0^+)dx \\ \int_0^l f(x)G(x \mid l^-)dx \end{pmatrix} \tag{1.34}$$

This process of writing the BIE at the two boundary points is called collocation.

By integrating the differential equation

$$T\frac{d^2}{dx^2}G(x \mid x_i) = -\delta(x - x_i)$$

we obtain $G(x \mid x_i)$ as

$$TG(x \mid x_i) = \begin{pmatrix} Ax + B; & x < x_i \\ Cx + D; & x > x_i \end{pmatrix}$$

$G(x \mid x_i)$ is continuous at $x = x_i$ and its derivative satisfies the jump condition of Eq.(1.16). These two conditions give us the relations

$$Ax_i + B = Cx_i + D \tag{1.35}$$

$$A - C = 1 \tag{1.36}$$

By utilizing Eqs.(1.35) and (1.36), we can write $G(x \mid x_i)$ as

$$TG(x \mid x_i) = \begin{pmatrix} x + Cx + B; \; x < x_i \\ x_i + Cx + B; \; x > x_i \end{pmatrix} \tag{1.37}$$

It is important to note that $G(x \mid x_i)$ contains two undetermined constants. These two constants can be determined by enforcing two arbitrary boundary conditions. Demanding that

$$G(x \mid x_i) = 0 \;\; at \;\; x = x_i - L \;\; and \;\; G(x \mid x_i) = 0 \;\; at \;\; x = x_i + L$$

we find

$$TG(x \mid x_i) = \begin{pmatrix} \frac{1}{2}(x - x_i) + \frac{L}{2}; \; x < x_i \\ \frac{1}{2}(x_i - x) + \frac{L}{2}; \; x > x_i \end{pmatrix} \tag{1.38}$$

The length L in Eq.(1.38) is arbitrary and this L has no effect on the final solution for $u(x)$. With $G(x \mid x_i)$ determined as in Eq.(1.38), we can write its derivative $F(x \mid x_i)$ as follows

$$TF(x \mid x_i) = \begin{pmatrix} \frac{1}{2}; \; x < x_i \\ -\frac{1}{2}; \; x > x_i \end{pmatrix} \tag{1.39}$$

We can now utilize Eqs.(1.39) and (1.40) to determine the coefficients of the matrices of Eq.(1.34).

$$TG(0 \mid 0^+) = \frac{L}{2}, \; TG(l \mid 0^+) = \frac{L-l}{2}, \; TG(0 \mid l^-) = \frac{L-l}{2}, \; TG(l \mid l^-) = \frac{L}{2} \tag{1.40}$$

$$TF(0 \mid 0^+) = \frac{1}{2}, \; TF(l \mid 0^+) = -\frac{1}{2}, \; TF(0 \mid l^-) = \frac{1}{2}, \; TF(l \mid l^-) = -\frac{1}{2} \tag{1.41}$$

The Eq.(1.34) now becomes

$$\begin{pmatrix} u(0^+) \\ u(l^-) \end{pmatrix} = \begin{bmatrix} -\frac{L}{2} & \frac{L-l}{2} \\ -\frac{L-l}{2} & \frac{L}{2} \end{bmatrix} \begin{pmatrix} q(0) \\ q(l) \end{pmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{pmatrix} u(0) \\ u(l) \end{pmatrix}$$

$$+ \frac{1}{T} \begin{pmatrix} \int_0^l f(x) \frac{L-x}{2} dx \\ \int_0^l f(x) \frac{x-l+L}{2} dx \end{pmatrix} \tag{1.42}$$

Rearranging Eq.(1.42), we obtain

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{pmatrix} u(0) \\ u(l) \end{pmatrix} - \begin{bmatrix} -\frac{L}{2} & \frac{L-l}{2} \\ -\frac{L-l}{2} & \frac{L}{2} \end{bmatrix} \begin{pmatrix} q(0) \\ q(l) \end{pmatrix}$$

$$= \frac{1}{T}\begin{pmatrix} \int_0^l f(x)\frac{L-x}{2}dx \\ \int_0^l f(x)\frac{x-l+L}{2}dx \end{pmatrix} \tag{1.43}$$

The advantage of Eq.(1.43) is that this equation can be employed to solve any type of boundary value problem for the differential equation

$$T\frac{d^2u}{dx^2} = -f(x)$$

We need not determine a new Green's functions when the nature of the boundary conditions changes. In that sense, the $G(x \mid x_i)$ of Eq.(1.38) is more general compared to the Green's function of Eqs.(1.13) or (1.31). We will call such a generalized solution of the accessory problem the fundamental solution.

Exercise 1.4: Show that

$$\frac{L}{2}[q(l) - q(0)] = -\frac{L}{2T}\int_0^l f(x)dx$$

and that the Eq.(1.43) can be collapsed into

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{pmatrix} u(0) \\ u(l) \end{pmatrix} - \begin{bmatrix} 0 & \frac{-l}{2} \\ \frac{l}{2} & 0 \end{bmatrix}\begin{pmatrix} q(0) \\ q(l) \end{pmatrix} = \frac{1}{T}\begin{pmatrix} -\int_0^l f(x)\frac{x}{2}dx \\ \int_0^l f(x)\frac{x-l}{2}dx \end{pmatrix} \tag{1.44}$$

This collapsing of Eq.(1.43) into Eq.(1.44) demonstrates how the arbitrary length L disappears from the final formulation of the problem.

In a boundary value problems, two out of the four quantities, $u(0), u(l), q(0), q(l),$ are given. Equation(1.44) is used to solve the remaining two unknowns. For example, if $u(0)$ and $q(l)$ are given, then we solve Eq.(1.44) for $q(0)$ and $u(l)$. In this manner, all four boundary quantities become known. In the next step, the BIE, Eq.(1.33), is used to calculate $u(x_i)$ at any desired location x_i.

Exercise 1.5:

For G and F, given in Eq.(1.38) and (1.39), respectively, show that the BIE of Eq.(1.33) becomes

$$u(x_i) = \frac{1}{2}u(0) + \frac{1}{2}u(l) + \frac{1}{2}x_i q(0) + \frac{1}{2}(x_i - l)q(l)$$

$$+ \frac{1}{2T}\int_0^{x_i}(x - x_i)f(x)dx + \frac{1}{2T}\int_{x_i}^l (x_i - x)f(x)dx \tag{1.45}$$

One particular combination of boundary conditions needs some special comments. When the gradients $q(0)$ and $q(l)$ are specified, Eq.(1.44) becomes,

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{pmatrix} u(0) \\ u(l) \end{pmatrix} = \frac{1}{T} \begin{pmatrix} -\int_0^l f(x)\frac{x}{2}dx \\ \int_0^l f(x)\frac{x-l}{2}dx \end{pmatrix} \tag{1.46}$$

In this situation, the coefficient matrix on the left-hand-side is singular and consequently one cannot solve for $u(0)$ and $u(l)$. Thus, in this situation, no solution of the problem can be obtained. This is not a drawback of the method, as it is well known that in this situation unique solution of the problem does not exist.

Example 1.6:
Solve

$$T\frac{d^2u}{dx^2} = -f(x)\ ;\ u(0) = a,\ q(l) = b$$

and write an expression for $u(x)$.
The Eq.(1.44) takes the form

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{pmatrix} a \\ u(l) \end{pmatrix} - \begin{bmatrix} 0 & \frac{-l}{2} \\ \frac{l}{2} & 0 \end{bmatrix} \begin{pmatrix} q(0) \\ b \end{pmatrix} = \frac{1}{T} \begin{pmatrix} -\int_0^l f(x)\frac{x}{2}dx \\ \int_0^l f(x)\frac{x-l}{2}dx \end{pmatrix}$$

These two equations yield

$$u(l) = a + lb + \frac{1}{T}\int_0^l xf(x)dx;\ q(0) = b + \frac{1}{T}\int_0^l f(x)dx$$

Substituting $u(0), u(l), q(0), q(l)$ in Eq.(1.45), we obtain

$$u(x_i) = a + bx_i + \frac{1}{T}\int_0^{x_i} xf(x)dx + \frac{1}{T}\int_{x_i}^l x_i f(x)dx$$

Compare this solution with the solution of Example 1.2.

Exercise 1.7:
Write the expression of $u(x_i)$ for the problem

$$\frac{d^2u}{dx^2} = 3x^2\ ;\ q(0) = 1,\ u(1) = 4$$

1.7 Adjoint Operator

In the previous sections, we developed a method, based on the fundamental solution, for the solution of the differentail equation

$$\frac{d^2u}{dx^2} = f(x)$$

If we wish to extend this technique to differentail equations of the form

$$\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu = f(x) \tag{1.47}$$

the idea of adjoint operator is essential. Using the identities

$$G\frac{d^2u}{dx^2} = \frac{d}{dx}\left(G\frac{du}{dx} - u\frac{dG}{dx}\right) + u\frac{d^2G}{dx^2}$$

and

$$aG\frac{du}{dx} = \frac{d}{dx}(aGu) - u\frac{d}{dx}(aG)$$

we can write

$$G\left[\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu\right] = \frac{d}{dx}\left(G\frac{du}{dx} - u\frac{dG}{dx}\right) + u\frac{d^2G}{dx^2} + \frac{d}{dx}(auG) - u\frac{d}{dx}(aG) + bGu$$

Rearranging this equation, we write

$$G\left[\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu\right] - u\left[\frac{d^2G}{dx^2} - a\frac{dG}{dx} + \left(b - \frac{da}{dx}\right)G\right]$$

$$= \frac{d}{dx}\left[G\frac{du}{dx} - u\frac{dG}{dx} + auG\right] \tag{1.48}$$

By defining a differential operator P as

$$P = \frac{d^2}{dx^2} + a\frac{d}{dx} + b \tag{1.49}$$

and an operator $\tilde{P}$ as

$$\tilde{P} = \frac{d^2}{dx^2} - a\frac{d}{dx} + \left(b - \frac{da}{dx}\right) \tag{1.50}$$

we can write Eq.(1.48) as

$$G(Pu) - u(\tilde{P}G) = \frac{d}{dx}B(u,q,G) \tag{1.51}$$

where the functional

$$B(u,q,G) = Gq - u\frac{dG}{dx} + auG \tag{1.52}$$

$B(u,q,G)$ is called the boundary functional.

The operator $\tilde{P}$ is known as the adjoint of P. The operators that satisfy $P = \tilde{P}$ are called self-adjoint.

In order to solve Eq.(1.47), one needs to obtain the fundamental solution by solving the equation

$$\widetilde{P}G(x \mid x_i) = \delta(x - x_i) \tag{1.53}$$

Noting that $Pu = f(x)$ and utilizing Eq.(1.53), we can write Eq.(1.51) as

$$G(x \mid x_i)f(x) - u(x)\delta(x - x_i) = \frac{d}{dx}B[u(x), G(x \mid x_i)] \tag{1.54}$$

Integrating Eq.(1.54) over the domain of the problem $x = 0$ to $x = l$, we get

$$u(x_i) = B[u(0), q(0), G(0 \mid x_i)] - B[u(l), q(l), G(l \mid x_i)] + \int_0^l G(x \mid x_i)f(x)dx \tag{1.55}$$

Equation (1.55) is the BIE for the problem of Eq.(1.47).

As stated before, $G(x \mid x_i)$ satisfies Eq.(1.53), is continuous at $x = x_i$, and satisfies the jump condition at $x = x_i$. The jump condition can be derived by integrating Eq.(1.53) between the limits x_i^- and x_i^+, as follows:

$$\int_{x_i^-}^{x_i^+} \frac{d^2G}{dx^2}dx - \int_{x_i^-}^{x_i^+} a\frac{dG}{dx}dx + \int_{x_i^-}^{x_i^+}\left(b - \frac{da}{dx}\right)Gdx = \int_{x_i^-}^{x_i^+} \delta(x - x_i)dx$$

By integrating the first term, intergrating the second term by parts, and integrating the right-hand-side, we can show that when b and da/dx are continuous

$$\frac{d}{dx}G(x \mid x_i)\ \Big|_{x=x_i^-} - \frac{d}{dx}G(x \mid x_i)\ \Big|_{x=x_i^+} = 1 \tag{1.56}$$

Example 1.8:

Convert the differential equation

$$\frac{d^2u}{dx^2} - 4\frac{du}{dx} - 5u = f(x)$$

into a BIE by determining a suitable fundamental solution.

For $a = -4$ and $b = -5$, the adjoint operator from Eq.(1.50) is

$$\widetilde{P} = \frac{d^2}{dx^2} + 4\frac{d}{dx} - 5$$

By solving

$$\frac{d^2G}{dx^2} + 4\frac{dG}{dx} - 5G = \delta(x - \xi)$$

we find

$$G(x \mid \xi) = \begin{pmatrix} Ae^{-5x} + Be^{x};\ x < \xi \\ Ce^{-5x} + De^{x};\ x > \xi \end{pmatrix}$$

The continuity and the jump condition, Eq.(1.56), yield

$$Ae^{-5\xi} + Be^{\xi} = Ce^{-5\xi} + De^{\xi} \tag{1.57}$$

$$-5Ce^{-5\xi} + De^{\xi} + 5Ae^{-5\xi} - Be^{\xi} = 1 \tag{1.58}$$

By solving Eqs.(1.57) and (1.58), we get

$$D = B + \frac{1}{6}e^{-\xi}; \quad A = C + \frac{1}{6}e^{5\xi}$$

Arbitrarily setting $B = 0$ and $C = 0$, we can write $G(x \mid x_i)$ as

$$G(x \mid \xi) = \begin{pmatrix} \frac{1}{6}e^{5(\xi-x)}; \ x < \xi \\ \frac{1}{6}e^{(x-\xi)}; \ x > \xi \end{pmatrix}$$

By defining

$$F(x \mid \xi) = \frac{dG}{dx} = \begin{pmatrix} -\frac{5}{6}e^{5(\xi-x)}; \ x < \xi \\ \frac{1}{6}e^{(x-\xi)}; \ x > \xi \end{pmatrix}$$

and $q = du/dx$, the functional B of Eq.(1.52) can now be written as

$$B(u,G) = Gq - Fu - 4uG$$

and the BIE of Eq.(1.55) can be written as

$$u(\xi) = \frac{1}{6}e^{5\xi}q(0) + \frac{1}{6}e^{5\xi}u(0) - \frac{1}{6}e^{(l-\xi)}q(l) + \frac{5}{6}e^{(l-\xi)}u(l)$$

$$+ \int_0^{\xi} \frac{1}{6}e^{5(\xi-x)}f(x)dx + \int_{\xi}^{l} \frac{1}{6}e^{(x-\xi)}f(x)dx \tag{1.59}$$

Example 1.9:

To write the collocation equations for the problem

$$\frac{d^2u}{dx^2} - 4\frac{du}{dx} - 5u = x\sin x; \quad 0 \le x \le 1$$

we set $\xi = 0$ and $\xi = 1$ in Eq.(1.59). The two equations that we obtain are:

$$\begin{bmatrix} \frac{5}{6} & -\frac{5}{6}e \\ -\frac{1}{6}e^5 & \frac{1}{6} \end{bmatrix} \begin{pmatrix} u(0) \\ u(1) \end{pmatrix} + \begin{bmatrix} -\frac{1}{6} & \frac{1}{6}e \\ -\frac{1}{6}e^5 & \frac{1}{6} \end{bmatrix} \begin{pmatrix} q(0) \\ q(1) \end{pmatrix}$$

$$= \begin{pmatrix} \int_0^1 \frac{1}{6}e^x x\sin x \, dx \\ \int_0^1 \frac{1}{6}e^{5(1-x)} x\sin x \, dx \end{pmatrix} \tag{1.60}$$

Among the four boundary values, $u(0), u(1), q(0),$ and $q(1),$ any two are specified and the two remaining boundary values are obtained by solving Eq.(1.60).

1.8 Gauss Quadrature

It is evident from the discussion in the previous sections that a key operation involved in the method of BIE is integration. Some of these intergrals may become too complex for analytical evaluation. We here introduce the method of Gauss quadrature for performing the complex integration tasks. For details of this method, readers should consult a text on numerical techniques. Here, we briefly describe the 6-point Gauss quadrature technique.

The integration of the function $z(\eta)$ over the range $\eta = -1$ to $\eta = +1$ is expressed as a finite sum over six terms as follows:

$$\int_{-1}^{+1} z(\eta) d\eta = \sum_{i=1}^{6} z(\eta_i) w_i \tag{1.60}$$

where

η_i	w_i
-0.9324 6951 4203	0.1713 2449 2379
-0.6612 0938 6466	0.3607 6157 3048
-0.2386 1918 6083	0.4679 1393 4572
0.2386 1918 6083	0.4679 1393 4572
0.6612 0938 6466	0.3607 6157 3048
0.9324 6951 4203	0.1713 2449 2379

In order to evaluate any general integral of the form

$$\int_a^b z(t) dt \tag{1.61}$$

one needs a transformation from t to η. Assuming a linear relationship as

$$t = \alpha\eta + \beta$$

we map the point $t = a$ to $\eta = -1$ and the point $t = b$ to $\eta = +1$. Thus,

$$a = -\alpha + \beta; \;\; b = \alpha + \beta$$

and

$$\beta = \frac{1}{2}(a+b); \ \alpha = \frac{1}{2}(b-a)$$

The integral of eq.(1.61) becomes

$$\int_a^b z(t)dt = \frac{1}{2}(b-a)\int_{-1}^{+1} z\left[\frac{1}{2}(b-a)\eta + \frac{1}{2}(a+b)\right]d\eta$$

$$= \sum_{i=1}^{6} w_i z\left[\frac{1}{2}(b-a)\eta_i + \frac{1}{2}(a+b)\right] \tag{1.62}$$

Exercise 1.10:
Solve the equation

$$\frac{d^2u}{dx^2} - 4\frac{du}{dx} - 5u = x\sin x \tag{1.63}$$

subjected to the boundary conditions $u(0) = 4$ and $q(1) = 0$

1.9 Conceptual Questions

At the end of this Chapter, you should be comfortable with the following concepts:

1. What is a Green's Function for a differential equation?
2. What is a delta function?
3. What are the properties of delta function?
4. How do you decide the boundary conditions for the Green's Function?
5. What is a fundamental solution of a differential equation?
6. What is the advantage of using the fundamental solution instead of using the Green's Function?
7. What is an adjoint operator?
8. What is the boundary fucntional?
9. How are the operator, the adjoint operator, and the boundary functional related?
10. How is the boundary integral equation derived?
11. What is collocation? Why is it necessary?
12. How can the existence of a solution be ensured from the collocation equations?
13. When all the boundary values are known, how is the boundary integral equation used?
14. Finite-Element or Finite-Difference methods give the solution at a set of nodes in the domain. How is the Boundary Element Method different from these methods, in this respect?
15. Why is the Boundary Element Method more efficient, at least for linear problems,

compared to the Finite-Element or the Finite-Difference methods? (think in terms of grid and the number of simultaneous equations that need to be solved)