Froudes' Momentum Theory: (Actuator Disk Theory)

Applications include Propellers,rotors and ducted fans.

Assumptions:

1. Infinitely thin disc of area A which offers no resistance to air passing through it.
2. Purely 1-D analysis
3. Thrust loading and velocity are uniform over the disk.
4. Far upstream and far down stream the pressure is freestream static pressure.
5. Viscous effects are not considered (no drag, no momentum diffusion)
6. Incompressible (compressibility correction can be made)

Figure 1: airflow

Consider an actuator disc at rest in a fluid which, is a long way ahead of the disc is moving uniformly with a speed of V_0 and has a pressure of P_0

Figure 2: side airflow

The outer curved lines represent the streamlines which seperate the fluid which passes through the disc. (A well defined slipstream.)

$$T = A(P_2 - P_1) \tag{1}$$

The increase in the rearward momentum of the air gives rise to thrust on the disk as a reaction.
Bernoulli's equation cannot be applied across the disk.
Bernoulli's constant is not the same across the disk.
Flow is divided into two regions 1 and 2 and Bernoulli's equation may be applied.

$$P_0 + 1/2\ \rho(V_0)^2 = P_1 + 1/2\ \rho(V_d)^2 \tag{2}$$

$$P_0 + 1/2\ \rho(V_e)^2 = P_2 + 1/2\ \rho(V_d)^2 \tag{3}$$

Now subtract 2 from 3 to get

$$1/2\ \rho(V_e^2 - V_0^2) = (P_2 - P_1) \tag{4}$$

$$T = A(P_2 - P_1) = A1/2\rho(V_e^2 - V_0^2) \tag{5}$$

$$T = \dot{m}(V_e - V_0) \tag{6}$$

$$T = \rho A V_d(V_e - V_0) \tag{7}$$

Equating 5 and 7

$\rho A/2(V_e^2 - V_0^2) = \rho A V_d(V_e - V_0)$

$(V_e + V_0)/2 = V_d$

$V_d = V_0(1 + a)$ where a is the inflow factor

$V_l + V_0 = 2V_d = 2V_0(1 + a)$

$V_e = V_0(1 + 2a)$

Rotor in climb

$$T = \rho A V_d(V_e - V_0) = 2\rho A V_0^2 a(1 + a) \tag{8}$$

$$V_d = V_0(1 + a) = V_0 + v \tag{9}$$

$$V_e = V_0(1 + 2a) = V_0 + 2v \tag{10}$$

Rotor in Hover

$V_0 = 0$
$V_d = v_h = v_i$
$V_e = 2v_h = 2v_i$
$T = 2\rho a v_h^2$
$v_h = \sqrt{T/(2\rho A)}$

Rotor thrust ,T, divided by the disc area A is called disc loading ,DL
$v_h = \sqrt{DL/2\rho}$

The higher the disc loading the stronger the induced velocity at the rotor (V_h) and the far wake velocity $2v_h$.

The induced velocity in the wake of a hovering rotor can produce operational problems if the hovering is done close to dust, sand, snow, or other loose surfaces.

Disadvantages:

1. lift dust, snow, or gravel which can be entrained in the wake and circulate through the rotor and engine intake.
2. cut off the pilots' view.

3. A high rotor downwash may make it difficult to work under a hovering helicopter while hooking up a sling load or guiding the pilot to a precision landing.

The higher the disc loading the more severe are the operational problems

Adavantages The higher DL permits compact helicopters with low empty weight –optimum for many applications.

Download on the fuselage:

The rotor wake contracts from the diameter of the rotor to its far wake size in about 1/4 to 1/2 of a rotor radius.

For most helicopters, the fuselage can be considered to be immersed in the remote wake and to receive the full effect of the downwash.

Vertical Drag (empirical)

$D_v = C_d q_{farwake} S = C_d(DL)S$

where C_d is the effective fuselage drag coefficient and S is the projected area of all affected components.

Ideal Power/ Induced Power

Ideal power is the power required to produce rotor thrust and is the true power supplied to the disc.

$P_i = TV_d = T(V_0 + V_e)/2$

For a hovering rotor $V_d = v_h$

Therfore $P_i = Tv_h = T\sqrt{T/(2\rho A)} = T^{2/3}/\sqrt{2\rho A}$

$P_i = T\sqrt{DL/2\rho}$

$T = \dot{m}w$

$P_i = Tv_i = \dot{m}w^2/2 = (2\rho A v_i^2)v_i = 2\rho A v_i^3$

$v_i = P_i/T = (PL)^{-1}$

For a given rotor thrust, the higher the disc loading, the higher the power required

In the early stages of the helicopter design DL was kept as low a 2 to 3lbs to minimize power required.

But the current treend is to have compact helicopters with minimum structrual weight. (largely due to lightweight turbine engines)

$DL = T/A$

For a given T:

DL ↓ implies → R–Rotor Radius↑

If DL ↓, A has to be increased.

Tail boom would have to be longer to achieve clearance between main and tail rotor.

Induced / Ideal power does not consider the viscous drag of the blades, namely the profile drag.

The distribution of the power losses of the rotor in hover :

1. Induced Power: 60%
2. Profile – viscous drag: 30%
3. Non-uniform inflow: 5 to 7%
4. Swirl in the wake: 1%
5. Tip losses 2 to 4%

<u>Figure of Merit: FM</u>

The ratio of the induced or ideal power to the actual power is known as the Figure of Merit.

FM = induced Power / Actual Power = TV_d/P\$actual.
For an ideal rotor FM = 1.0
Very good practical rotor FM = 0.8
<u>For Axial Flight:</u>

$$\eta_{axial} = \frac{T\frac{V_e+V_0}{2}}{P_{actual}} = \frac{V_0+V}{P_{actual}}$$

$FM = \frac{P_{ideal}}{P_{actual}} quit$ or $P_{actual} = \frac{P_{ideal}}{FM}$

Figure 3: Loading

For a given T and FM, higher DL implies lower power loading and higher power required.

But lower DL → higher PL and threfore less power required.

Reduction of Power required by lowering disc loading would mean added strructual weight and a larger overall size with perhaps little increase in payload capacity.

Induced power is the lower bound on the power required.

Nondimensional Coefficients

Thrust Coefficient:
$C_T = \frac{T}{\rho A(\Omega R)^2} = \frac{T}{\rho A V_T^2}$

Torque Coefficient:
$C_Q = \frac{Q}{\rho A(\Omega R)^2 R}$

Power Coefficient:
$C_P = \frac{P}{\rho A(\Omega R)^3}$

Torque and Power Coefficients are numerically equal.
$P = Q\Omega$
$C_P = \frac{Q\Omega}{\rho A(\Omega R)^3} = \frac{Q}{\rho A(\Omega R)^2 R} = C_Q$

Rotor Solidity Ration σ

$\frac{total\ blade\ area}{Disc\ area} = \sigma = \frac{N_b C R}{\pi R^2} = \frac{N_b C}{\pi R}$

For hovering rotor

$\lambda_i = \frac{V_i}{\Omega R} = \frac{1}{\Omega R}\sqrt{\frac{T}{2\rho A}} = \sqrt{\frac{T}{2\rho A(\Omega R)^2}} = \sqrt{\frac{C_T}{2}}$
$C_{pi} = TV_i = C_T^{3/2}/\sqrt{2}$
$C_{pi} = \frac{kC_T^{3/2}}{\sqrt{2}}$
$FM = \frac{P_{ideal}}{P_{actual}} = \frac{C_T^{3/2}/\sqrt{2}}{kC_T^{3/2}/\sqrt{2}+1/8\sigma C_{do}}$ $C_{po} = \frac{1}{8}\sigma C_{do}$

Profile Power
$dP = d\vec{Q} * \vec{\Omega}$
($\vec{Q} and \vec{\Omega}$ are along and axis perpendicular to v)
$= N_b(dD * yDy)\Omega$
$P_{profile} = N_b\Omega \int_0^R dDydy$
$= N_b\Omega \int 0^R [\frac{1}{2}C_d\rho(V_d)^2 c]ydy$
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$V \approx V_t \approx \Omega y$ $V_T \approx \Omega$ $C_d = C_{do}$
$P_0 = P_{profile} = N_b\Omega \int 0^R \frac{1}{2}Cdo\rho(\Omega y)^2 cydy$

$P_0 = [N_b \frac{\Omega^3}{2} C_{do}\rho c] \int 0^R y^3 dy$
$= \frac{1}{8} N_b \Omega^3 C_{do} R^4$
$\frac{P_0}{\rho A(\Omega R)^3} = \frac{1}{8}\frac{N_b C R}{A}C_{do} = \frac{1}{8}\frac{N_b C R}{piR^2}C_{do} = \frac{1}{8}\frac{N_b C}{piR}C_{do} = \frac{1}{8}\sigma C_{do}$

Power Loading (PL) $\frac{T}{P}$ in hover.

$\frac{T^{3/2}}{\sqrt{2\rho A}} = \frac{T}{2\rho A}$

Figure 4: Profile Power

$\frac{P}{T} = \frac{1}{T}\frac{T}{2\rho A} = v_i = (PL)^{-1}$
$\frac{P}{T} = k(\frac{T}{2\rho A})^{\frac{1}{2}} + \frac{P_0}{T}$
$= \Omega R \frac{C_{P\ actual}}{C_T}$
$= \frac{\Omega R}{C_T}[k\frac{C_T^{3/2}}{\sqrt{2}} + \frac{1}{8}\sigma C_{do}]$

Axial Translation (Climb)
$T = \dot{m}(V_c + w) - \dot{m}V_c$
$T = PAV_d(V_e - V_o)$
$= \rho AV_d(2v)$

$$T = \rho A(V_o + v)(2v) \tag{11}$$

$$P = TV_d = 2\rho A(V_0 + v)v \tag{12}$$

From Thrust: 11
$T = 2\rho A(V_0 v + v^2)$
or
$v^2 + V_0 v - \frac{T}{2\rho A} = 0$
$v = \frac{-V_0 \overset{+}{-} \sqrt{V_0^2 + 4\frac{T}{2\ rhoA}}}{2}$
For Climb
$v = \frac{-V_0}{2} + \sqrt{\frac{V_0^2}{4} + \frac{T}{2\rho A}}$
$\frac{T}{2\rho A} = V_h$
Given DL or T and A and the climb velocity we can calculate induced velocity at the disc.
$\frac{V_i}{V_h} = \frac{-V_0}{2V_h} + \sqrt{(\frac{V_0}{2V_h})^2 + 1}$
$\frac{P_i}{P_{actual}} = FM$ axial Transmission losses $P_{actual}*$
$P_i =$ ideal Power $= TV_d$
$= 2\rho A(V_0 + v)^2 v$

$= 2\rho A(V_0^2 + 2V_0 V + v^2)v$

For a given V_0, P_i, ρ,A

$f(v) = v^3 + 2v^2 V_0 + vV_0^2 - \frac{P_i}{2\rho A} = 0$

$f'(v) = 3v^2 + 4vV_0 + V^2$

$\frac{f_1 - f_0}{v_1^* - V_0} = f_1'$

$f_0 = 0$

$V_1 - \frac{f_1}{f_1'} = v$

Figure 5: graph

Where a is the inflow factor

$T = \rho A V_d (V_e - V_0)$

$= \rho A V_0 (1 + a)(V_0(1 + 2a) - V_0)$

$T = \rho A V_0 (1 + a)(V_0 2a) = 2\rho A V_0^2 (a + a^2)$

$V_d = V_0(1 + a)$

If we call $V_d = V_0 + V_0 A = V + w$

$V_e = V_0(1 + 2a)$

or $V_0 a = w = (V_0 + 2w$

Where w is the induced velocity

$$\underline{T = \rho A(V_0 + w)}$$

Efficiency $= \frac{work\ done/unit\ time}{work\ input/unit\ time}$

Power supplied by the disc to the fluid equals the difference in the flux of kinetic energy (through the stream tube)

Reacall $\rho A V_d = \dot{m}$

$P_r = \dot{m}(\frac{V_e^2 - V^2}{2}$

$= \rho A V_d (V_e - V)(\frac{V_e + V}{2})$

$P_r = TV_d$ ideal power)

Power (usefull obtained) TV

$\eta_i deal = \frac{V_d}{V_d} = \frac{2V}{V_e + V}$

$eta_i = \frac{V}{V_d} = \frac{V}{V_d(1+a)} = \frac{1}{1+a}$
$V_a = u$
$\eta_i = \frac{1}{1+w/v}$
$\eta_i = \frac{V}{V_d} = \frac{2V}{V_e+V} = \frac{2}{1+\frac{V_e}{V}}$
$\frac{V_e}{V}$ should be close to unity to have high efficiency or a should be zero
Thrust produced $T = \rho A V_d (V_e - V)$
$\rho A V_d (V(1+2a) - V)$
$\rho A V_d (2va)$