Simplification to BE theory

Without simplification: (Climb or hover)

$$V_R^2 = \left[(V_0+v_i)^2 + r(\Omega-\omega)^2\right] = (\Omega r)^2\left[\left(\frac{V_0+v_i}{\Omega r}\right)^2 + (1-a_1)^2\right]$$

Assume no swirl $\rightarrow a_1 = 0$.
Assume $|\Omega r| >> |V_0 + v_i|$.

$$\frac{V_0+v_i}{\Omega r} \sim 0$$
$$V_R = \Omega r$$

Recall

$$dT = BC\frac{1}{2}\rho V_R^2(C_l\cos\phi - Cd\sin\phi)dr$$

Using small angle approximation ($\sin\phi \simeq \phi$, $\cos\phi \simeq 1$)

$$dT \simeq dL \simeq BC\frac{1}{2}\rho V_R^2 C_l dr$$

$C_d\phi << C_l$ as C_d and ϕ are small quantities.
Recall

$$dQ = BC(\frac{1}{2}\rho V_R^2)(C_l\sin\phi + C_d\cos\phi)rdr$$
$$dQ = BC(\frac{1}{2}\rho V_R^2)(C_l\phi + C_d)rdr$$

Where $C = C(r)$, $C_l = C_l(r,\alpha,Re,M,\dot{\alpha})$, and $C_d = C_d(r,\alpha,Re,M,\dot{\alpha})$.

$$\frac{dT}{\rho(\Omega r)^2\pi R^2} = dC_T = \frac{BC}{\rho(\Omega r)^2\pi R^2}\frac{1}{2}\rho V_R^2 C_l dr = \frac{BC}{\pi R}\frac{1}{2}\left(\frac{r}{R}\right)^2 C_l d(\frac{r}{R})$$
$$dC_T = \frac{1}{2}\sigma_s(\bar{r})^2 C_l d(\bar{r})$$
$$C_T = \frac{1}{2}\int_0^1 \sigma_s(\bar{r})^2 C_l d(\bar{r})$$

Where $\bar{r} = r/R$, $\sigma_s = \frac{BC}{\pi R}$, and $C = C(\bar{r})$.

$$\frac{dQ}{\rho(\Omega r)^2\pi R^2 R} = dC_Q = \frac{BC}{\rho(\Omega r)^2\pi R^2 R}\frac{1}{2}\rho V_R^2(C_l\phi + C_d)rdr = \frac{BC}{\pi R}\frac{1}{2}\left(\frac{r}{R}\right)^2(C_l\phi + C_d)\frac{r}{R}d(\frac{r}{R})$$
$$dC_Q = \frac{1}{2}\sigma_s(\bar{r})^3(C_l\phi + C_d)d(\bar{r})$$
$$C_Q = \frac{1}{2}\int_0^1 \sigma_s(\bar{r})^3(C_l\phi + C_d)d(\bar{r})$$

Where

$$\phi = \tan^{-1}\left[\frac{V_o+v_i}{r(\Omega-\omega)}\right] \simeq \left[\frac{V_0+v_i}{r(\Omega-\omega}\right] = \frac{U_P}{U_T}$$

$\lambda =$ Axial inflow ration $= \frac{U_P}{\Omega R}$
U_P is velocity normal to the tip path plane. When the rotor is in forward flight with no coning and no flapping

$$\lambda = \frac{V_0\sin\alpha + v_i}{\Omega R}$$

In axial translation ($\alpha = 90°$)

$$\lambda = \frac{V_0 + v_i}{\Omega R}$$

$$\phi \simeq \frac{U_P}{U_T} \simeq \frac{V_0 + v_i}{\Omega r} = \left(\frac{V_0 + v_i}{\Omega R}\right)\left(\frac{r}{R}\right) = \frac{\lambda}{\bar{r}}$$

Approximation ($C_l = C_{l\alpha}(\alpha - \alpha_{L0})$)
Where α_{L0} is an angle of attack at zero lift.

$$\begin{aligned} C_l &\simeq C_{l\alpha}\alpha \simeq C_{l\alpha}(\theta - \phi) \\ C_T &= \frac{1}{2}\int_0^1 \sigma_s C_{l\alpha}(\theta - \phi)(\bar{r})^2 d(\bar{r}) \\ &= \frac{1}{2}\int_0^1 \sigma_s C_{l\alpha}(\theta - \frac{\lambda}{\bar{r}})(\bar{r})^2 d(\bar{r}) \\ &= \frac{1}{2}\int_0^1 \sigma_s C_{l\alpha}(\theta\bar{r}^2 - \lambda\bar{r}) d(\bar{r}) \end{aligned}$$

Possible approximation for θ

- Uniform pitch: $\theta = \theta_0 = const.$
- Linear twist: $\theta = \theta_0 + \theta_{tw}\bar{r}$ or $\theta = \theta_{0.75} + \theta_{tw}(\bar{r} - 3/4)$
- Ideal twist: $\theta = \theta_{tw}\bar{r}$

Other approximations

- Constant chord (C is a constant)
- Uniform inflow (λ is a constant)

$$\begin{aligned} dC_T &= \frac{1}{2}\sigma_s(\bar{r})^2 C_l d(\bar{r}) \\ dC_Q &= \frac{1}{2}\sigma_s(\bar{r})^3(C_l\phi + C_d)d(\bar{r}) \\ &= \frac{1}{2}\sigma_s(\bar{r})^3(C_l\phi)d(\bar{r}) + \frac{1}{2}\sigma_s(\bar{r})^3(C_d)d(\bar{r}) \\ &= dC_T\phi\bar{r} + \frac{1}{2}\sigma_s(\bar{r})^3(C_d)d(\bar{r}) \\ &= dC_T\lambda + \frac{1}{2}\sigma_s(\bar{r})^3(C_d)d(\bar{r}) \\ C_Q &= \int_0^1 \left[dC_T\lambda + \frac{1}{2}\sigma_s(\bar{r})^3(C_d)d(\bar{r})\right] \end{aligned}$$

If λ is a constant (uniform inflow) and ($C_d = C_{do}$)

$$C_Q = C_T\lambda + \frac{\sigma}{8}C_{do}$$

Figure of Merit

$$FM = \frac{\text{Induced power}}{\text{Actual power}} = \frac{\int (dT)v_i}{\int (dQ)\Omega} = \frac{\int (dT)v_i}{\int (dQ)\Omega}\frac{\rho(\Omega R)^3(\pi R^2)}{\rho(\Omega R)^3(\pi R^2)}$$
$$= \frac{\int dC_T\left(\frac{v_i}{\Omega R}\right)}{\int dC_Q} = \frac{\int dC_T\lambda}{\int dC_Q}$$

Neglecting swirl $dC_Q = dC_T\lambda + \frac{1}{2}\sigma_s(\bar{r})^3(C_d)d(\bar{r})$

$$FM = \frac{\int dC_T\lambda}{\int dC_T\lambda + \frac{1}{2}\sigma_s(\bar{r})^3(C_d)d(\bar{r})}$$

If we assume uniform inflow $\lambda = const.$

$$FM = \frac{C_T\lambda}{C_T\lambda + \frac{1}{2}\int \sigma_s(\bar{r})^3(C_d)d(\bar{r})}$$

Constant chord ($\sigma_s = \sigma$)

$$FM = \frac{C_T\lambda}{C_T\lambda + \frac{1}{2}\sigma\int(\bar{r})^3(C_d)d(\bar{r})}$$

If $C_d = C_{do} = const.$

$$FM = \frac{C_T\lambda}{C_T\lambda + \frac{1}{2}\sigma C_{do}\int(\bar{r})^3 d(\bar{r})}$$

Integrating from $\bar{r} = 0$ to 1 we get

$$FM = \frac{C_T\lambda}{C_T\lambda + \frac{\sigma C_{do}}{8}}$$

In hover for uniform inflow momentum theory gives $\lambda = \sqrt{\frac{C_T}{2}}$

$$FM = \frac{C_T^{3/2}/\sqrt{2}}{C_T^{3/2}/\sqrt{2} + \frac{\sigma C_{do}}{8}}$$

To account for other losses a factor k is added to give

$$FM = \frac{C_T^{3/2}/\sqrt{2}}{kC_T^{3/2}/\sqrt{2} + \frac{\sigma C_{do}}{8}}$$

Where k is 1.0 to 1.15
The integration limits do not always go from $\bar{r} = 0$ to 1.0. Parameters such as root cut-out and tip loss factor can change the limits.