Special cases of thrust constant equations

$\alpha = 0$ (Hover like orientation)
$\alpha = 90$ (Propeller or climb)

Hover $\alpha = 0$ Thrust is aligned with the z-axis.
Consider the general expression for induced velocity

$$\left(\frac{v_i}{v_h}\right)^4 + 2\left(\frac{v_i}{v_h}\right)^3\left(\frac{V_0}{v_h}\right)\sin\alpha + \left(\frac{v_i}{v_h}\right)^2\left(\frac{V_0}{v_h}\right)^2 - 1 = 0$$

$\sin\alpha = 0$ The induced velocity equation is a biquadratic

$$\left(\frac{v_i}{v_h}\right)^4 + \left(\frac{v_i}{v_h}\right)^2\left(\frac{V_0}{v_h}\right)^2 - 1 = 0$$

or

$$\left(\frac{v_i}{v_h}\right)^2 = -\frac{1}{2}\left(\frac{V_0}{v_h}\right)^2 + \sqrt{\frac{1}{4}\left(\frac{V_0}{v_h}\right)^4 + 1}$$

or

$$\frac{v_i}{v_h} = \left[-\frac{1}{2}\left(\frac{V_0}{v_h}\right)^2 + \sqrt{\frac{1}{4}\left(\frac{V_0}{v_h}\right)^4 + 1}\right]^{\frac{1}{2}}$$

Climb/propeller $\alpha = 90$

$$T = \rho A(V_0 + v_i)(2v_i)$$
$$\frac{T}{2\rho A} = V_0 v_i + v_i^2$$
$$v_h^2 = V_0 v_i + v_i^2$$
$$\left(\frac{v_i}{v_h}\right)^2 + \left(\frac{v_i}{v_h}\right)\left(\frac{V_0}{v_h}\right) - 1 = 0$$

But $\frac{T}{2\rho A} = v_h$ for the rotor hovering at the same thrust. Solving for $\frac{v_i}{v_h}$ yields

$$\frac{v_i}{v_h} = -\frac{1}{2}\left(\frac{V_0}{v_h}\right) + \sqrt{\frac{1}{4}\left(\frac{v_i}{v_h}\right)^2 + 1}$$

Constant power equations

Variation of v_i and T with forward velocity V_0 for constant power. In hover the thrust that can be developed at a given power

$$P_{i,h} = T_h v_h \qquad (A)$$

With the same power the rotor at an angle of attack α in a freestream velocity V_0 produces a T and induced velocity v_i given by the equation

$$P_i = P_{i,h} = T(V_0 \sin\alpha + v_i) \qquad (B)$$

Dividing (B) by (A) we get

$$\frac{P_i}{P_{i,h}} = \frac{P_{i,h}}{P_{i,h}} = 1 = \frac{T}{T_h}\left(\frac{V_0\sin\alpha + v_i}{v_h}\right)$$

or

$$\frac{T}{T_h} = \left[\frac{V_0\sin\alpha + v_i}{v_h}\right]^{-1}$$

For a hovering rotor

$$
\begin{aligned}
P_{i,h} &= T_h v_h \\
v_h &= \left(\frac{T_h}{2\rho A}\right)^{\frac{1}{2}} \\
v_h &= \left(\frac{P_{i,h}}{2\rho A}\right)^{\frac{1}{3}} \\
P_{i,h} &= T_h \left(\frac{T_h}{2\rho A}\right)^{\frac{1}{2}} \\
P_{i,h} &= T_h^{\frac{3}{2}} \left(\frac{1}{2\rho A}\right)^{\frac{1}{2}} \\
T_h^{\frac{3}{2}} &= P_{i,h} = (2\rho A)^{\frac{1}{2}} \\
T_h &= P_{i,h}^{\frac{2}{3}} (2\rho A)^{\frac{1}{3}}
\end{aligned}
$$

From Glauerts' hypothesis

$$T = \rho A V_d (2 v_i)$$

Where $V_d^2 = [V_0 \cos\alpha)^2 + (v_i + V_0 \sin\alpha)^2]$

$$\left(\frac{T}{2\rho A}\right)^2 = (V_0^2 + 2V_0 v_i \sin\alpha + v_i^2) v_i^2$$

Dividing by $\left(\frac{T_h}{2\rho A}\right)^2 = v_h^4$ we get

$$
\begin{aligned}
\left(\frac{T}{T_h}\right)^2 &= \left(\frac{v_i}{v_h}\right)^4 + 2\left(\frac{V_0}{v_h}\right)\left(\frac{v_i}{v_h}\right)^3 \sin\alpha + \left(\frac{V_0}{v_h}\right)^2 \left(\frac{v_i}{v_h}\right)^2 \\
1 &= \left[\left(\frac{v_i}{v_h}\right)^4 + 2\left(\frac{V_0}{v_h}\right)\left(\frac{v_i}{v_h}\right)^3 \sin\alpha + \left(\frac{V_0}{v_h}\right)^2 \left(\frac{v_i}{v_h}\right)^2\right] \left(\frac{T_h}{T}\right)^2
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{T}{T_h} &= \left[\frac{V_0 \sin\alpha + v_i}{v_h}\right]^{-1} \\
\frac{T_h}{T} &= \left[\frac{V_0 \sin\alpha + v_i}{v_h}\right]
\end{aligned}
$$

$$\left[\left(\frac{v_i}{v_h}\right)^4 + 2\left(\frac{V_0}{v_h}\right)\left(\frac{v_i}{v_h}\right)^3 \sin\alpha + \left(\frac{V_0}{v_h}\right)^2 \left(\frac{v_i}{v_h}\right)^2\right] \left[\frac{V_0 \sin\alpha + v_i}{v_h}\right]^2 - 1 = 0$$

Solve this equation for given V_0, P, and α. For constant power calculations

1. $v_h = (P_{i,h}/2\rho A)^{\frac{1}{3}}$
2. For V_0/v_h and α solve

   $$\left[\left(\frac{v_i}{v_h}\right)^4 + 2\left(\frac{V_0}{v_h}\right)\left(\frac{v_i}{v_h}\right)^3 \sin\alpha + \left(\frac{V_0}{v_h}\right)^2 \left(\frac{v_i}{v_h}\right)^2\right] \left[\frac{V_0 \sin\alpha + v_i}{v_h}\right]^2 - 1 = 0$$

   to obtain $\frac{v_i}{v_h}$ from which we get
3. $\frac{T}{T_h} = \left[\frac{V_0 \sin\alpha + v_i}{v_h}\right]^{-1}$