Induced velocity and thrust in non-axial translation

The thrust developed by a rotor moving along its axis at a speed V is equal to the rate of mass flow through the disc times the doubled induced velocity at the disc. In this case, the rate of mass flow is clearly defined as the product of the disc area ($A = \pi R^2$) times the air density ρ, times the resultant flow through the disc: $\vec{V}_d = \vec{V}_0 + \vec{v}_i$ which in axial translation is identical to the algebraic sum.

The rotor makes an angle α with the x-axis. Therefore the freestream $\vec{V}_0$ makes an angle $(90 - \alpha)$ with the thrust vector. The resultant velocity through the rotor is

$$\vec{V}_d = \vec{V}_0 + \vec{v}_i$$

Far downstream the velocity is postulated to be $\vec{V}_0 + 2\vec{v}_i$ where $\vec{v}_i$ is the induced velocity also along the same direction as thrust but in the opposite sense.

For axial translation Glauert hypothesis states

$$\begin{aligned} T &= \dot{m}\Delta V \\ &= \rho A V_d (V_e - V_0) \\ T &= 2\rho A V_d v_i \end{aligned}$$

For non-axial translation a similar analysis is used.

Forward Flight (Glauert)

Glauerts hypothesis $\begin{cases} \vec{V}_d = \vec{V}_0 + \vec{v}_i \\ \vec{V}_e = \vec{V}_0 + 2\vec{v}_i \end{cases}$

$$\begin{aligned} T &= \dot{m}(\Delta V) \\ &= \rho A V_d (V_e - V_0) \\ &= \rho A V_d (2v_i) \\ V_d &= \left[(V_0 \cos\alpha)^2 + (v_i + V_0 \sin\alpha)^2\right]^{\frac{1}{2}} \\ T &= 2\rho A \left[(V_0 \cos\alpha)^2 + (v_i + V_0 \sin\alpha)^2\right]^{\frac{1}{2}} v_i \\ \left(\frac{T}{2\rho A}\right)^2 &= (V_0^2 + 2V_0 v_i \sin\alpha + v_i^2) v_i^2 \qquad eqn.(A) \end{aligned}$$

When the rotor is hovering with the same thrust

$$T = 2\rho A V_d (V_e - V_0) = 2\rho A v_h^2 \quad \text{or} \quad v_h = \left(\frac{T}{2\rho A}\right)^{\frac{1}{2}} \qquad eqn.(B)$$

Substitute eqn.(B) in eqn.(A) to get

$$v_h^4 = V_0^2 v_i^2 + 2V_0 v_i^3 \sin\alpha + v_i^4$$

Divide by v_h^4 and rearrange to get

$$\left(\frac{v_i}{v_h}\right)^4 + 2\left(\frac{v_i}{v_h}\right)^3 \left(\frac{V_0}{v_h}\right) \sin\alpha + \left(\frac{v_i}{v_h}\right)^2 \left(\frac{V_0}{v_h}\right)^2 - 1 = 0$$

Power required ($= P_{ideal} = P_i$)

$$\begin{aligned}
P_i &= \frac{1}{2}\dot{m}(V_e^2 - V_0^2) \\
&= \frac{1}{2}\dot{m}\left[(V_0 \sin\alpha + 2v_i)^2 + (V_0\cos\alpha)^2 - V_0^2\right] \\
&= \frac{1}{2}\dot{m}\left[V_0^2 \sin^2\alpha + 4v_i V_0 \sin\alpha + 4v_i^2 + V_0^2\cos^2\alpha - V_0^2\right] \\
&= \frac{1}{2}\dot{m}\left[4v_i(V_0 \sin\alpha + v_i)\right] \\
&= \dot{m}(2v_i)(V_0 \sin\alpha + v_i) \\
&= T(V_0 \sin\alpha + v_i)
\end{aligned}$$

Ideal power required by the rotor is given by the product of the thrust and velocity normal to the disc. If we denoted the idea (induced) power required to hover and produce the same thrust $P_{i,h}$. Then

$$\begin{aligned}
P_{i,h} &= Tv_h \\
\frac{P_i}{P_{i,h}} &= \frac{T(V_0 \sin\alpha + v_i)}{Tv_h} = \left(\frac{V_0}{v_h}\right)\sin\alpha + \left(\frac{v_i}{v_h}\right)
\end{aligned}$$

<u>Methods</u>

Given T,α, and V_0 calculate P_i solve

$$v_i^4 + 2V_0 \sin\alpha v_i^3 + V_0^2 v_i^2 - \left(\frac{T}{2\rho A}\right)^2 = 0$$

to obtain v_i and then calculate P_i from

$$P_i = T(V_0 \sin\alpha + v_i)$$

This also provided in the form of charts by solving

$$\left(\frac{v_i}{v_h}\right)^4 + 2\left(\frac{v_i}{v_h}\right)^3\left(\frac{V_0}{v_h}\right)\sin\alpha + \left(\frac{v_i}{v_h}\right)^2\left(\frac{V_0}{v_h}\right)^2 - 1 = 0$$

and

$$\frac{P_i}{P_{i,h}} = \left(\frac{V_0}{v_h}\right)\sin\alpha + \left(\frac{v_i}{v_h}\right)$$

sequentially.