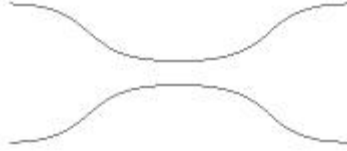
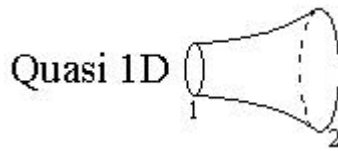


Quasi-1D Nozzle Review

Example: Want to find P,T,M, etc. given P_o , P_e , and nozzle shape.



Quasi – Area is allowed to vary along x coordinate, but flow variables are functions of x only.



Start out with governing conservation equations:

Mass:

$$\iiint_V \frac{\partial \rho}{\partial t} dV + \iint_S \rho \bar{U} \cdot \bar{n} dS = 0 \quad (1.1)$$

Momentum:

$$\frac{\partial}{\partial t} \iiint_V \rho \bar{U} dV + \iint_S \rho (\bar{U} \cdot \bar{n}) \bar{U} dS = - \iint_S p dS + \iiint_V \rho \bar{f} dV + F_{viscous} \quad (1.2)$$

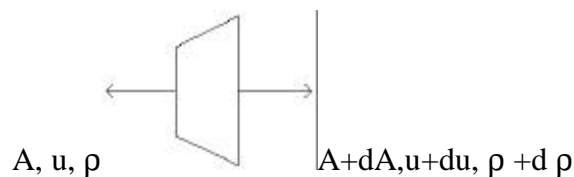
Energy:

$$\frac{\partial}{\partial t} \iiint_V \rho \left(e + \frac{U^2}{2} \right) dV + \iint_S \rho \left(e + \frac{U^2}{2} \right) \bar{U} \cdot \bar{n} dS = - \iint_S p \bar{U} \cdot \bar{n} dS + \iiint_V \rho \frac{\partial q}{\partial t} dV + \iiint_V \rho (\bar{f} \cdot \bar{U}) dV \quad (1.3)$$

Assuming:

1. steady
2. inviscid
3. no body forces
4. 2D flow

Quasi-1D: Area is allowed to vary but flow variables are a function of x only



Mass:

$$-ruA + (u + du)(r + dr)(A + dA) = 0 \quad (1.4)$$

$$-\cancel{ruA} + \cancel{ruA} + rudA + rduA + d ruA + \text{higher order terms} = 0 \quad (1.5)$$

divide by through ruA

$$\boxed{\frac{dA}{A} + \frac{du}{u} + \frac{dr}{r} = 0}$$

or

$$\boxed{d(ruA) = 0} \quad (1.6)$$

Momentum:

$$-ru^2A + (r + dr)(u + du)(u + du)(A + dA) = PA - (P + dP)(A + dA) + 2\left(\frac{PdA}{2}\right) \quad (1.7)$$

$$-\cancel{ru^2A} + \cancel{ru^2A} + ru^2dA + u^2Adr + ruAdu + ruAdu = \cancel{PA} - \cancel{PA} - \cancel{PdA} - AdP + \cancel{PdA} \quad (1.8)$$

$$u(\cancel{rudA} + \cancel{uAdr} + \cancel{rAdu}) + ruAdu = -AdP \quad (1.9)$$

$$dP = -rudu \quad (1.10)$$

$$\frac{dP}{r} = \frac{dP}{dr} \frac{dr}{r} = -udu \left[a^2 = \frac{dP}{dr} \right] \quad (1.11)$$

$$\frac{dr}{r} = -\frac{u}{a^2} du \quad (1.12)$$

Substituting equation (1.12) into (1.6) to get:

$$\frac{dA}{A} + \frac{du}{u} - \frac{u}{a^2} du = 0 \quad (1.13)$$

which can be rearranged to get:

$$\frac{dA}{A} + \frac{du}{u} \left(1 - \frac{u^2}{a^2} \right) = \frac{dA}{A} + \frac{du}{u} (1 - M^2) = 0 \quad (1.14)$$

$$\boxed{\frac{dA}{A} = \frac{du}{u} (M^2 - 1)} \quad (1.15)$$

which can be used to determine general flow behavior in a converging-diverging nozzle, as below:

	M	dA	du
subsonic	< 1	< 0	> 0
	< 1	> 0	< 0
supersonic	> 1	< 0	< 0
	> 1	> 0	> 0

dA < 0 --> converging
dA > 0 --> diverging
du < 0 --> decelerating
du > 0 --> accelerating

Energy:

We will not go through the derivation for the energy equation but,

applying analysis as before will give:

$$dh + udu = 0 \quad \text{or} \quad c_p dT = -udu \quad (1.16)$$

Total-Static-Mach Relations

Isentropic Relations:

$$\frac{P_2}{P_1} = \left(\frac{r_2}{r_1} \right)^{\frac{\mathbf{g}}{\mathbf{g}-1}} = \left(\frac{T_2}{T_1} \right)^{\frac{\mathbf{g}}{\mathbf{g}-1}} \quad (1.17)$$

Static-Total

From Energy Equation with $u_o=0$ (total)

$$c_p T_o = c_p T_1 + \frac{u_1^2}{2} \quad (1.18)$$

Rearranging

$$\frac{T_o}{T} = 1 + \frac{u^2}{2c_p T} \quad (1.19)$$

using $c_p = \frac{\mathbf{g}R}{\mathbf{g}-1}$,

$$\frac{T_o}{T} = 1 + \frac{\mathbf{g}-1}{2} \frac{u^2}{\mathbf{g}RT} \quad (1.20)$$

Inserting $a = \sqrt{\mathbf{g}RT}$ and $M = u/a$ gives

$$\frac{T_o}{T} = 1 + \frac{\mathbf{g}-1}{2} M^2 \quad (1.21)$$

$$\frac{P_o}{P} = \left(1 + \frac{\mathbf{g}-1}{2} M^2 \right)^{\frac{\mathbf{g}}{\mathbf{g}-1}} \quad (1.22)$$

$$\frac{r_o}{r} = \left(1 + \frac{\mathbf{g}-1}{2} M^2 \right)^{\frac{1}{\mathbf{g}-1}} \quad (1.23)$$

Given Eq.'s (1.21),(1.22), and (1.23) we can now:

- Find any static property in an isentropic flow given Mach #, P_o, T_o, ρ_o .
- Use/control known total conditions to find mach # through nozzle

Area-Mach Relations

From mass

$$r^* u^* A^* = r u A \quad (1.24)$$

at $A^* M = \frac{u^*}{a^*} = 1 \rightarrow u^* = a^*$, giving

$$\left(\frac{A}{A^*}\right)^2 = \left(\frac{r^*}{r}\right)^2 \left(\frac{a^*}{u}\right)^2 \quad (1.25)$$

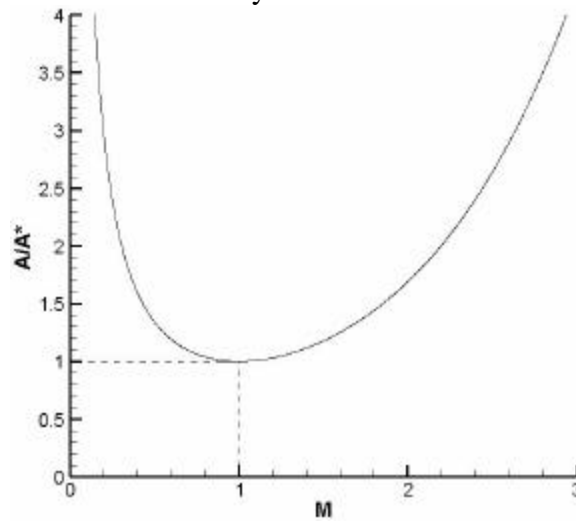
or

$$\left(\frac{A}{A^*}\right)^2 = \left(\frac{r^*}{r_o}\right)^2 \left(\frac{r_o}{r}\right)^2 \left(\frac{a^*}{u}\right)^2 \quad (1.26)$$

using isentropic relations for the density terms

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2} \left[\frac{2}{g+1} \left(1 + \frac{g-1}{2} M^2 \right) \right]^{\frac{g+1}{g-1}} \quad (1.27)$$

Mach # is a function of this area ratio only. Must find A^* .



Completely Subsonic Flow

$$P_e = P_{atm} \quad (1.28)$$

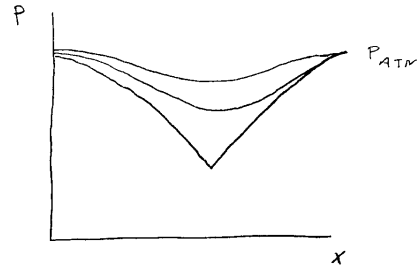
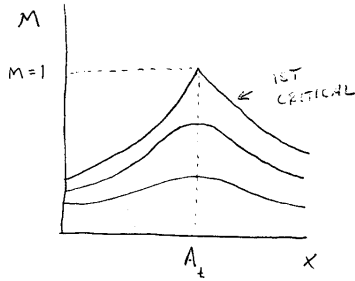
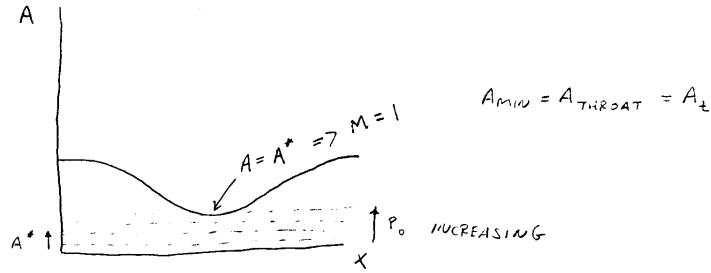
From isentropic relation

$$M_e = \sqrt{\frac{2}{g-1} \left[\left(\frac{P_o}{P_e} \right)^{\frac{g-1}{g}} - 1 \right]} \quad (1.29)$$

Can now determine A^* and entire Mach # distribution

If: $\frac{P_o}{P_e} = 1$ Then: $M_e = 0$, $\frac{A_e}{A^*} \rightarrow \infty$, $A^* = 0$ NO FLOW

As $\frac{P_o}{P_e}$ increases, M_e increases, $\frac{A_e}{A^*}$ decreases, A^* increases



Note that $A/A^* < 1$ is not physically possible. That is, after 1st critical is reached, must have $A_{min} = A^*$

Supersonic Flow

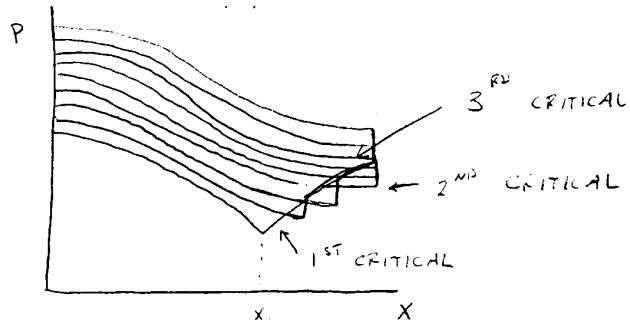
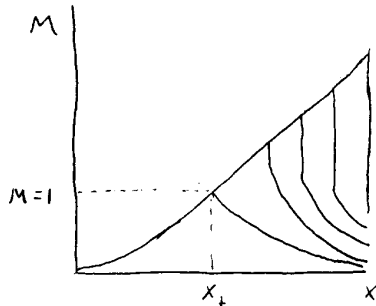
Subsonic Flow ahead of throat. Follow supersonic A/A^* branch after throat.

$$A^* = A_t \tag{1.30}$$

$$M_e = f\left(\frac{A_e}{A^*}\right) \tag{1.31}$$

$$P_e = P_o \left(1 + \frac{\gamma-1}{2} M_e^2\right)^{\frac{\gamma}{1-\gamma}} \tag{1.32}$$

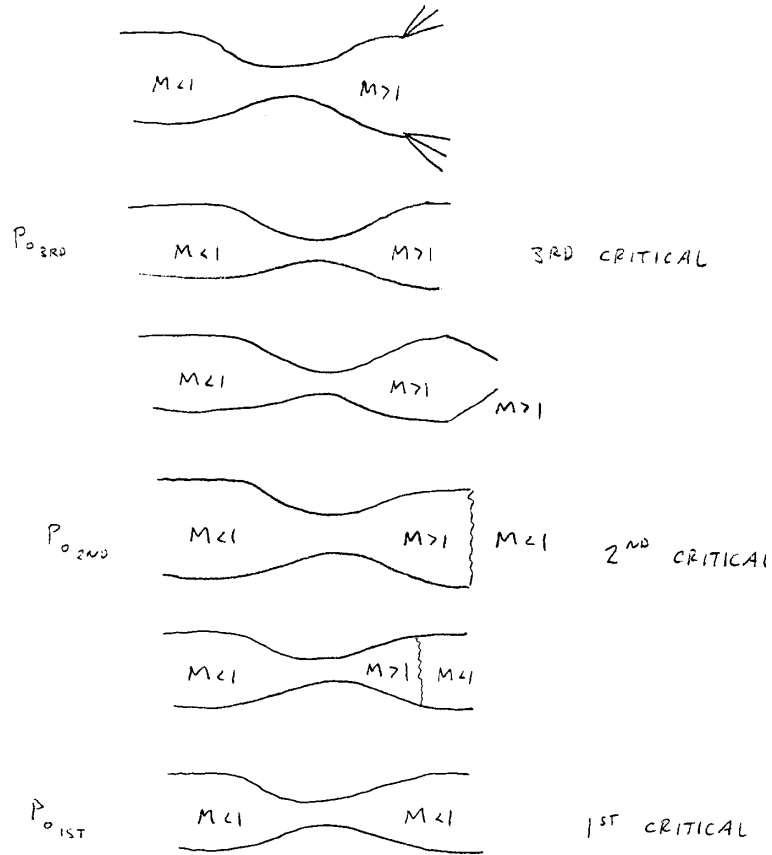
Suppose we pick P_o so that $P_e = P_{atm}$



- If we decrease P_o , then $P_e < P_{atm}$ because M_e is unchanged. Need weak oblique shocks to get a small pressure jump.
- As P_o decreases, need stronger oblique shocks until normal shock at exit, 2nd critical.

- As P_o decreases, shock moves up the nozzle. Eventually get to 1st critical.
- Increasing P_o from 3rd critical, $P_e > P_{atm}$. Get Prandtl-Meyer expansion fan to get pressure decrease

Summary:



For our nozzle:

$$P_{o,3rd} \approx 60 \text{ psig}$$

$$P_{o,2nd} \approx 20 \text{ psig}$$

$$P_{o,1st} \approx 8 \text{ psig}$$