

1. For each of the following, write the equations in a standard form, and identify the surface in space they describe. You don't need to graph them.

(a)  $441x^2 + 144y^2 + 784z^2 - 7056 = 0$ .

First, add 7056 to both sides and get  $441x^2 + 144y^2 + 784z^2 = 7056$ . Divide both sides by 7056 to obtain  $\frac{x^2}{16} + \frac{y^2}{49} + \frac{z^2}{9} = 1$ . This is the equation of an ellipsoid with  $x$ -radius = 4,  $y$ -radius = 7, and  $z$ -radius = 3.

(b)  $7x - 4y^2 + 3z^2 = 0$

Notice that the  $x$  is not raised to the second power. So we'll isolate it. Adding  $4y^2 - 3z^2$  to both sides yields  $7x = 4y^2 - 3z^2$ . Then divide both sides by 7 to get  $x = \frac{4}{7}y^2 - \frac{3}{7}z^2$ . The standard form of this equation is  $x = \frac{y^2}{(7/4)} - \frac{z^2}{(7/3)}$ . This is the equation of a hyperbolic paraboloid.

(c)  $x^2 + z^2 - 8x + 4z + 13 = 0$

Here, we have two variables not raised to the second power. This is a tipoff to apply the method of completing the square. Grouping  $x$  and  $z$  terms together on the left side and subtracting 13 from both sides yields  $(x^2 - 8x) + (z^2 + 4z) = -13$ . Now complete the square as follows:

$(x^2 - 8x + 16) + (z^2 + 4z + 4) = -13 + 16 + 4 \Rightarrow (x - 4)^2 + (z + 2)^2 = 7$ . This is the equation of a right circular cylinder whose central axis is the line  $x = 4$ ,  $z = -2$  and has radius  $\sqrt{7}$ .

(d)  $2x^2 + 2y^2 + 2z^2 - 12x - 6y + 28z + 81 = 0$

Here we have three variables not raised to the second power. So once again, we resort to completing the square. But first, we must divide both sides by 2 and get,  $x^2 + y^2 + z^2 - 6x - 3y + 14z + \frac{81}{2} = 0$ .

Grouping  $x$ ,  $y$ , and  $z$  terms together on the left side and subtracting  $\frac{81}{2}$  from both sides yields

$$(x^2 - 6x) + (y^2 - 3y) + (z^2 + 14z) = -\frac{81}{2}. \text{ Now complete the square to get}$$

$$(x^2 - 6x + 9) + \left(y^2 - 3y + \frac{9}{4}\right) + (z^2 + 14z + 49) = -\frac{81}{2} + 9 + \frac{9}{4} + 49$$

$$\Rightarrow (x - 3)^2 + \left(y - \frac{3}{2}\right)^2 + (z + 7)^2 = \frac{-162 + 36 + 9 + 196}{4}$$

$$\Rightarrow (x - 3)^2 + \left(y - \frac{3}{2}\right)^2 + (z + 7)^2 = \frac{79}{4}.$$

This is the equation of a sphere centered at  $\left(3, \frac{3}{2}, -7\right)$  of radius  $\frac{\sqrt{79}}{2}$ .

2. Let  $f(x, y) = -8x^2 + 48x - 2y^2 - 8y + 20$ .

- (a) Identify and write the equation for each of the level curves of this function for  $z = k$ , where  $k = 92, 68, 28, -28$ .

Set  $z = f(x, y)$ . First, let's do this for  $z = k$  in general. Then we look at  $k = -8x^2 + 48x - 2y^2 - 8y + 20$ . By completing the square, we get the following:

$$k - 20 = -8(x^2 - 6x) - 2(y^2 + 4y) \Rightarrow k - 8(9) - 2(4) - 20 = -8(x^2 - 6x + 9) - 2(y^2 + 4y + 4) \\ \Rightarrow k - 100 = -8(x - 3)^2 - 2(y + 2)^2.$$

If  $k = 92$ , we then have  $-8 = -8(x - 3)^2 - 2(y + 2)^2 \Rightarrow (x - 3)^2 + \frac{(y + 2)^2}{4} = 1$ . This is an ellipse centered at  $(3, -2)$  with vertical radius 2 and horizontal radius 1. If  $k = 68$ , we then have

$-32 = -8(x - 3)^2 - 2(y + 2)^2 \Rightarrow \frac{(x - 3)^2}{4} + \frac{(y + 2)^2}{16} = 1$ . This is an ellipse centered at  $(3, -2)$  with vertical radius 4 and horizontal radius 2. If  $k = 28$ , we then have  $-72 = -8(x - 3)^2 - 2(y + 2)^2 \Rightarrow$

$\frac{(x - 3)^2}{9} + \frac{(y + 2)^2}{36} = 1$ . This is an ellipse centered at  $(3, -2)$  with vertical radius 6 and horizontal radius 3. If  $k = -28$ , we then have  $-128 = -8(x - 3)^2 - 2(y + 2)^2 \Rightarrow \frac{(x - 3)^2}{16} + \frac{(y + 2)^2}{64} = 1$ . This

is an ellipse centered at  $(3, -2)$  with vertical radius 8 and horizontal radius 4.

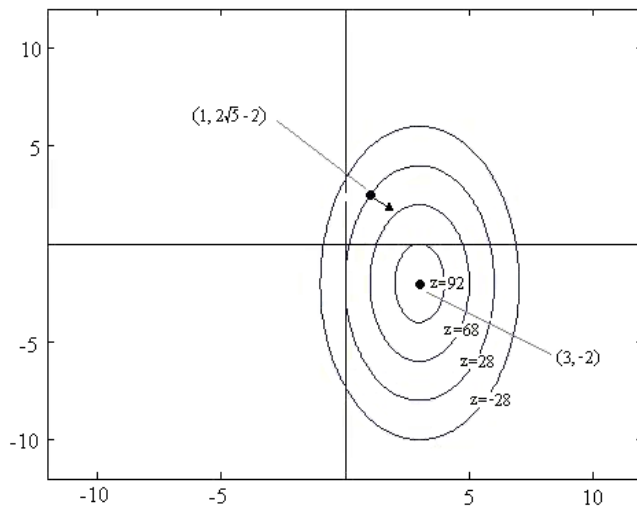
- (b) Consider the point  $\vec{p} = \langle 1, 2(\sqrt{5} - 1) \rangle$ . Determine which level curve from part (a) that  $\vec{p}$  lies on. Then find the unit vector giving the direction of most rapid increase in  $f$  at  $\vec{p}$ .

First plug  $x = 1$  and  $y = 2\sqrt{5} - 2$  into the equation  $k - 100 = -8(x - 3)^2 - 2(y + 2)^2$ . Then we get  $k - 100 = -8(1 - 3)^2 - 2(2\sqrt{5})^2 = -8(4) - 2(20) = -32 - 40 = -72$ . Then we solve

for  $k$  and get  $k = 28$ . So  $\vec{p}$  lies on the level curve  $\frac{(x - 3)^2}{9} + \frac{(y + 2)^2}{36} = 1$  for  $f$ . To compute the direction of greatest increase, we compute  $\nabla f(\vec{p})$ .  $\nabla f = \langle -16x + 48, -4y - 8 \rangle$ , and thus  $\nabla f(\vec{p}) = \langle -16(1) + 48, -4(2\sqrt{5} - 2) - 8 \rangle = \langle 32, -8\sqrt{5} + 8 - 8 \rangle = \langle 32, -8\sqrt{5} \rangle$ .

The unit vector is  $\frac{\langle 32, -8\sqrt{5} \rangle}{|\langle 32, -8\sqrt{5} \rangle|} = \frac{\langle 32, -8\sqrt{5} \rangle}{\sqrt{1024 + 320}} = \frac{\langle 32, -8\sqrt{5} \rangle}{\sqrt{1344}} = \frac{\langle 32, -8\sqrt{5} \rangle}{8\sqrt{21}} = \left\langle \frac{4}{\sqrt{21}}, -\sqrt{\frac{5}{21}} \right\rangle \approx \langle .872872, -.48795 \rangle$ .

- (c) Sketch the level curves you found in part (a), and the unit vector from part (b) emanating from  $(1, 2\sqrt{5} - 2)$  on the axis provided, making sure to label each curve with the corresponding  $z = k$ .



3. Let  $f(x, y) = \begin{cases} \frac{x^2 - 7xy + 12y^2}{x - 3y} & \text{if } x \neq 3y \\ g(x) & \text{if } x = 3y \end{cases}$ . If  $f$  is continuous on the whole plane, find  $g(x)$ .
- If  $x \neq 3y$ ,  $f(x, y) = \frac{x^2 - 7xy + 12y^2}{x - 3y} = \frac{(x - 3y)(x - 4y)}{x - 3y} = x - 4y$ . When  $x = 3y$ ,  $y = \frac{x}{3}$ . So we want
- $$g(x) = x - 4\left(\frac{x}{3}\right) = -\frac{1}{3}x.$$

4. Find the following limits, or state that they do not exist.

(a) 
$$\lim_{(x,y) \rightarrow (3,-2)} \frac{2x^2 - 7xy - 15y^2}{2x + 3y}.$$

$$\lim_{(x,y) \rightarrow (3,-2)} \frac{2x^2 - 7xy - 15y^2}{2x + 3y} = \lim_{(x,y) \rightarrow (3,-2)} \frac{(2x + 3y)(x - 5y)}{2x + 3y}$$

$$= \lim_{(x,y) \rightarrow (3,-2)} x - 5y = 3 - 5(-2) = 13.$$

(b) 
$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3y - xy}{x^2(x^2 - 1)^2 + y^2}.$$
 (Suggestion: Look at the cubic equations  $y = x^3 - x$  and  $y = -x^3 + x$ .)

Along the path  $y = x^3 - x$ ,  $\frac{x^3y - xy}{x^2(x^2 - 1)^2 + y^2} = \frac{x^3(x^3 - x) - x(x^3 - x)}{(x^3 - x)^2 + (x^3 - x)^2} = \frac{(x^3 - x)^2}{2(x^3 - x)^2} = \frac{1}{2}$ . There-

fore,  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3y - xy}{x^2(x^2 - 1)^2 + y^2} = \frac{1}{2}$  along the curve  $y = x^3 - x$ .

Similarly, along the path  $y = -x^3 + x$ ,  $\frac{x^3y - xy}{x^2(x^2 - 1)^2 + y^2} = \frac{x^3(-x^3 + x) - x(-x^3 + x)}{(x^3 - x)^2 + (-x^3 + x)^2}$   

$$= \frac{x^3(-x^3 + x) - x(-x^3 + x)}{(x^3 - x)^2 + (-1)^2(x^3 - x)^2} = \frac{-(x^3 - x)^2}{2(x^3 - x)^2} = -\frac{1}{2}.$$
 Therefore,  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3y - xy}{x^2(x^2 - 1)^2 + y^2} = -\frac{1}{2}$  along the curve  $y = -x^3 + x$ .

Since the limit is different along 2 distinct paths through  $(0, 0)$ , then  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3y - xy}{x^2(x^2 - 1)^2 + y^2}$  does not exist.

5. Let  $f(x, y) = 4 - x^2 - y^2$ . Give the parametric equations for the line tangent to the curve of intersection between the surface  $z = f(x, y)$  and the plane  $y = x$  at the point  $(1, 1, 2)$ .

Since the plane  $y = x$  will contain the tangent line, we want to compute the derivative of  $f$  in the direction  $\vec{i} + \vec{j}$ . So the unit direction vector is  $\vec{u} = \frac{\langle 1, 1 \rangle}{\sqrt{1+1}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ . Next, we find the directional derivative  $D_{\vec{u}}f(\vec{p}) = \vec{u} \cdot \nabla f(\vec{p})$ , where  $\vec{p} = \langle 1, 1 \rangle$ .  $\nabla f = \langle -2x, -2y \rangle$ , and so  $\nabla f(1, 1) = \langle -2, -2 \rangle$ . Hence  $D_{\vec{u}}f(\vec{p}) = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \cdot \langle -2, -2 \rangle = -\sqrt{2} - \sqrt{2} = -2\sqrt{2}$ . This tells us that for each unit we go in the direction  $\vec{i} + \vec{j}$ ,  $z$  decreases by  $2\sqrt{2}$ . But a one unit increase in the direction of  $\vec{i} + \vec{j}$  is an increase in  $x$  by  $\frac{1}{\sqrt{2}}$  units and an increase in  $y$  by  $\frac{1}{\sqrt{2}}$  units. Therefore, for every increase of  $\frac{1}{\sqrt{2}}$  units in the  $x$  and  $y$  directions,  $z$  decreases  $2\sqrt{2}$  units. So the direction of the tangent line is  $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -2\sqrt{2} \right\rangle$ , or  $\langle 1, 1, -4 \rangle$  (multiplying all 3 components by  $\sqrt{2}$  to simplify matters). Therefore, we obtain the parametric equations for the tangent line:

$$x = 1 + t, y = 1 + t, z = 2 - 4t.$$

6. Consider the surface defined by the equation  $xy^2 + 3x - z^2 = 4$ .

- (a) Find the equation of the tangent plane to the surface at the point  $(2, 1, -2)$ .

Set  $F(x, y, z) = xy^2 + 3x - z^2 - 4$ . The normal to the tangent plane is given by  $\nabla F(2, 1, -2)$ .  $\nabla F = \langle y^2 + 3, 2xy, -2z \rangle$ , and so  $\nabla F(2, 1, -2) = \langle (1)^2 + 3, 2(2)(1), -2(-2) \rangle = \langle 4, 4, 4 \rangle$ . Therefore, the equation of the tangent plane is given by  $4(x - 2) + 4(y - 1) + 4(z + 2) = 0 \Rightarrow 4x + 4y + 4z = 4 \Rightarrow x + y + z = 1$ .

- (b) Give the symmetric equations for the line perpendicular to the surface at  $(2, 1, -2)$ .

We've already done the required work. Just plug in the coordinates of the point and the direction numbers to obtain the symmetric equations:

$$\frac{x - 2}{4} = \frac{y - 1}{4} = \frac{z + 2}{4}$$

7. Find the indicated derivatives.

(a)  $\frac{\partial f}{\partial y}$  if  $f(w, x, y, z) = \frac{e^{xy} \cos(w^2 z^2)}{wxyz}$ .

Here, we don't need chain rule since  $w, x, y,$  and  $z$  are all independent variables. Using the quotient rule, we obtain the following:

$$f_y = \frac{(wxyz) \cos(w^2 z^2) (xe^{xy}) - e^{xy} \cos(w^2 z^2)(wxz)}{(wxyz)^2} = \frac{wxze^{xy} \cos(w^2 z^2)}{(wxyz)^2} [xy - 1] = \frac{(xy - 1)e^{xy} \cos(w^2 z^2)}{wxy^2z}.$$

(b)  $\frac{dz}{dt}$  if  $z = \sqrt{x^2 + y^2 + 4}, x = t^2 + 3t, y = 2t + 3$ .

Here,  $z$  depends on  $x$  and  $y$ , but  $x$  and  $y$  both depend on  $t$ . So by the chain rule (first form),

$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$ . From what we're given we compute the following:

$$\frac{\partial z}{\partial x} = \frac{1}{2}(x^2 + y^2 + 4)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2 + 4}},$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(x^2 + y^2 + 4)^{-1/2}(2y) = \frac{y}{\sqrt{x^2 + y^2 + 4}},$$

$\frac{dx}{dt} = 2t + 3$ , and  $\frac{dy}{dt} = 2$ . Then we have

$$\begin{aligned} \frac{dz}{dt} &= \frac{x}{\sqrt{x^2 + y^2 + 4}}(2t + 3) + \frac{y}{\sqrt{x^2 + y^2 + 4}}(2) = \frac{(2t + 3)x + 2y}{\sqrt{x^2 + y^2 + 4}} = \frac{(2t + 3)(t^2 + 3t) + 2(2t + 3)}{\sqrt{(t^2 + 3t)^2 + (2t + 3)^2 + 4}} \\ &= \frac{(2t + 3)[t^2 + 3t + 2]}{\sqrt{t^4 + 6t^3 + 9t^2 + 4t^2 + 12t + 9 + 4}} = \frac{(2t + 3)(t^2 + 3t + 2)}{\sqrt{t^4 + 6t^3 + 13t^2 + 12t + 13}}. \end{aligned}$$

(c)  $\frac{\partial w}{\partial s}$  if  $w = x^2y + 3x^3 + 4y^2, x = 2st,$  and  $y = 3s - t$ .

$w$  depends on two variables,  $x$  and  $y$ , but both of these depend on the two variables  $s$  and  $t$ . So by the chain rule (second form) we have  $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{dx}{ds} + \frac{\partial w}{\partial y} \frac{dy}{ds}$ . From what we have, we compute the

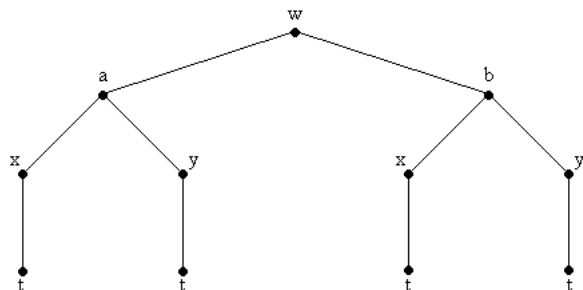
following:

$$\frac{\partial w}{\partial x} = 2xy + 9x^2, \quad \frac{\partial w}{\partial y} = x^2 + 8y$$

$\frac{dx}{ds} = 2t$ , and  $\frac{dy}{ds} = 3$ . Therefore,

$$\begin{aligned} \frac{\partial w}{\partial s} &= (2xy + 9x^2)(2t) + (x^2 + 8y)(3) = [2(2st)(3s - t) + 9(2st)^2](2t) + [(2st)^2 + 8(3s - t)](3) \\ &= [2(6s^2t - 2st^2) + 9(4s^2t^2)](2t) + [4s^2t^2 + 24s - 8t](3) \\ &= [12s^2t - 4st^2 + 36s^2t^2](2t) + [12s^2t^2 + 72s - 24t] \\ &= 24s^2t^2 - 8st^3 + 72s^2t^3 + 12s^2t^2 + 72s - 24t \\ &= 36s^2t^2 - 8st^3 + 72s^2t^3 + 72s - 24t. \end{aligned}$$

8. Let  $w = a^2b + 2a - b$ ,  $a = x + y$ ,  $b = x - y$ ,  $x = 2t$ , and  $y = t + 3$ . Write a chain rule to find  $\frac{dw}{dt}$ . You do not need to calculate the derivative. The dependent variable is  $w$ , and the independent variable is  $t$ .  $w$  depends on  $a$  and  $b$ , but  $a$  and  $b$  depend on  $x$  and  $y$ . Furthermore  $x$  and  $y$  depend on  $t$ . So we draw the following schematic diagram.



So the appropriate chain rule goes as follows:

$$\frac{dw}{dt} = \frac{\partial w}{\partial a} \left[ \frac{\partial a}{\partial x} \frac{dx}{dt} + \frac{\partial a}{\partial y} \frac{dy}{dt} \right] + \frac{\partial w}{\partial b} \left[ \frac{\partial b}{\partial x} \frac{dx}{dt} + \frac{\partial b}{\partial y} \frac{dy}{dt} \right].$$

9. Let  $f(x, y) = 9 - x^2 - y^2$ .

- (a) Evaluate  $f(1, 2)$  and  $f(1.05, 2.1)$  to calculate  $\Delta z$ .

$f(1, 2) = 9 - (1)^2 - (2)^2 = 4$  and  $f(1.05, 2.1) = 9 - (1.05)^2 - (2.1)^2 = 9 - 1.1025 - 4.41 = 3.4875$ . Recall that  $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$ . For this particular problem, we have  $x = 1$ ,  $y = 2$ ,  $\Delta x = 0.05$ , and  $\Delta y = 0.1$ . Hence,  $\Delta z = f(1.05, 2.1) - f(1, 2) = 3.4875 - 4 = -0.5125$ .

- (b) Find the total differential  $dz$ .

The formula for  $dz$  is  $dz = f_x(x, y)\Delta x + f_y(x, y)\Delta y$ .  $f_x(x, y) = -2x$  and  $f_y(x, y) = -2y$ . Therefore,  $dz = -2x\Delta x - 2y\Delta y$ . With  $x = 1$ ,  $y = 2$ ,  $\Delta x = 0.05$ , and  $\Delta y = 0.1$ , we have  $dz = -2(1)(0.05) - 2(2)(0.1) = -0.1 - 0.4 = -0.5$ .

10. Use the methods from Section 12.8 to find the point on the plane  $3x + 2y + z = 13$  that is closest to  $(1, 1, 1)$ , and use the Second Partials Test to verify that the point is in fact a minimum.

If  $(x, y, z)$  is an arbitrary point, its distance from  $(1, 1, 1)$  is  $\sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2}$ . For convenience, we will minimize the distance squared. So  $d^2 = (x-1)^2 + (y-1)^2 + (z-1)^2$ . But the point must be on the surface  $3x + 2y + z = 13$ . So solve this for  $z$  and get  $z = 13 - 3x - 2y$ . Plug into  $d^2$  and get  $d^2 = (x-1)^2 + (y-1)^2 + (13 - 3x - 2y - 1)^2 \Rightarrow d^2 = (x-1)^2 + (y-1)^2 + (12 - 3x - 2y)^2$ . Now set  $f(x, y) = (x-1)^2 + (y-1)^2 + (12 - 3x - 2y)^2$ . We first find the critical points of  $f$ . Since  $f$  is a polynomial in  $x$  and  $y$  and there are no boundaries, the only critical points are stationary ones. So we find the gradient  $\nabla f(x, y)$  and set it equal to  $\langle 0, 0 \rangle$ .  $f_x = 2(x-1) + 2(12 - 3x - 2y)(-3) = 2x - 2 - 72 + 18x + 12y = 20x + 12y - 74$ . Similarly  $f_y = 2(y-1) + 2(12 - 3x - 2y)(-2) = 2y - 2 - 48 + 12x + 8y = 12x + 10y - 50$ . So setting  $\nabla f(x, y) = \langle 0, 0 \rangle$  leads to the linear system of equations

$$\begin{cases} 20x + 12y = 74 \\ 12x + 10y = 50 \end{cases} \Rightarrow \begin{cases} 10x + 6y = 37 \\ 6x + 5y = 25 \end{cases}$$

Multiply the top equation by  $-3$  and the bottom equation by  $5$  to eliminate  $x$  and get the system

$$\begin{cases} -30x - 18y = -111 \\ 30x + 25y = 125 \end{cases}$$

Add, then get  $7y = 14$ . Therefore,  $y = 2$ . Plug back into  $6x + 5y = 25$  and get  $6x + 10 = 25$

$$\Rightarrow 6x = 15 \Rightarrow x = \frac{15}{6} = \frac{5}{2}. \text{ So the critical point in } (x, y) \text{ is } \left(\frac{5}{2}, 2\right).$$

Next, we use the Second Partials Test to verify that this is actually a minimum. We already have  $f_x$  and  $f_y$ . From these, we compute  $f_{xx} = 20$ ,  $f_{yy} = 10$ , and  $f_{xy} = 12$ . Then for all  $(x, y)$ , we get  $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y) = (20)(10) - 12^2 = 200 - 144 = 56 > 0$ ,

Also, for any  $(x, y)$ ,  $f_{xx} = 20 > 0$ . We certainly have  $D\left(\frac{5}{2}, 2\right) > 0$  and  $f_{xx}\left(\frac{5}{2}, 2\right) > 0$ . So by the

Second Partials Test, the point  $\left(\frac{5}{2}, 2\right)$  is a minimum of  $f(x, y)$ .

Finally, we find the  $z$ -coordinate. The easiest way is to use  $z = 13 - 3x - 2y$ . Therefore,

$$z = 13 - 3\left(\frac{5}{2}\right) - 2(2) = 9 - \frac{15}{2} = \frac{3}{2}. \text{ Therefore, the point closest to the point } (1, 1, 1) \text{ on the plane}$$

$$3x + 2y + z = 13 \text{ is } \left(\frac{5}{2}, 2, \frac{3}{2}\right).$$

11. Use the method of Lagrange multipliers to find the point on the plane  $3x + 2y + z = 13$  that is closest to  $(1, 1, 1)$ .

Once again, we will use the trick of maximizing the distance squared. So let  $f(x, y, z) = (x - 1)^2 + (y - 1)^2 + (z - 1)^2$ . Since we are minimizing the distance from the plane, the equation of the plane is a constraint. So set  $g(x, y, z) = 3x + 2y + z - 13 = 0$ . Next, we set up the system

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x, y, z) = 0 \end{cases} . \text{ We have } f_x = 2(x - 1) = 2x - 2, f_y = 2(y - 1) = 2y - 2, \text{ and } f_z = 2(z - 1) = 2z - 2.$$

Also,  $g_x = 3$ ,  $g_y = 2$ , and  $g_z = 1$ . Therefore, our system becomes

$$\begin{cases} 2x - 2 = 3\lambda \\ 2y - 2 = 2\lambda \\ 2z - 2 = \lambda \\ 3x + 2y + z - 13 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{3\lambda}{2} + 1 \\ y = \lambda + 1 \\ z = \frac{\lambda}{2} + 1 \\ 3x + 2y + z = 13 \end{cases} .$$

Plugging the top three equations into the bottom one yields

$$3\left(\frac{3\lambda}{2} + 1\right) + 2(\lambda + 1) + \left(\frac{\lambda}{2} + 1\right) = 13 \Rightarrow \frac{9\lambda}{2} + 3 + 2\lambda + 2 + \frac{\lambda}{2} + 1 = 13 \\ \Rightarrow 7\lambda + 6 = 13 \Rightarrow 7\lambda = 7 \Rightarrow \lambda = 1.$$

Plugging  $\lambda = 1$  into the first three equations yield  $x = \frac{5}{2}$ ,  $y = 2$ , and  $z = \frac{3}{2}$ . Therefore, the point

$\left(\frac{5}{2}, 2, \frac{3}{2}\right)$  is the point on the plane  $3x + 2y + z = 13$  that is closest to  $(1, 1, 1)$ .

12. (Follow-up to questions 10 and 11) What is the minimum distance from the plane  $3x + 2y + z = 13$  to the point  $(1, 1, 1)$ ?

The minimum distance is the distance from  $\left(\frac{5}{2}, 2, \frac{3}{2}\right)$  to  $(1, 1, 1)$ , which is

$$\sqrt{\left(\frac{5}{2} - 1\right)^2 + (2 - 1)^2 + \left(\frac{3}{2} - 1\right)^2} = \sqrt{\frac{9}{4} + 1 + \frac{1}{4}} = \sqrt{\frac{7}{2}} \approx 1.87083.$$