

Integral Formulas from Chapter 14:

Line Integrals: Section 14.2

1.) If C is a piecewise smooth curve in the plane, and it has the parametric equations $x = x(t)$; $y = y(t)$; $a \leq t \leq b$, then

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2} dt.$$

2.) If C is a piecewise smooth curve in space, and it has the parametric equations $x = x(t)$; $y = y(t)$; $z = z(t)$; $a \leq t \leq b$, then

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt.$$

Remark:

C given by the parametric equations $x = x(t)$; $y = y(t)$; $a \leq t \leq b$ is equivalent to C being defined by the curvilinear motion $\mathbf{r}(t) = \langle x(t), y(t) \rangle$; $a \leq t \leq b$. Similarly, C given by the parametric equations $x = x(t)$; $y = y(t)$; $z = z(t)$; $a \leq t \leq b$ is equivalent to C being defined by the curvilinear motion $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$; $a \leq t \leq b$. So both of the above integrals can be written as

$$\int_C f(\mathbf{r}) ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{v}(t)\| dt.$$

Work Integrals: Section 14.2

1.) Let $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ be a force field acting on a point $Q = (x, y)$ in the plane. The amount of work required to move Q along a curve C in the plane is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Unravel the meaning of this expression: If C has the parametric equations $x = x(t)$; $y = y(t)$; $a \leq t \leq b$, then C is the graph of the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. So $\mathbf{F} \cdot d\mathbf{r} = \langle M, N \rangle \cdot \langle dx, dy \rangle = M dx + N dy$. Hence we can write

$$W = \int_C [M dx + N dy].$$

2.) Let $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a force field acting on a point $Q = (x, y, z)$ in space. The amount of work required to move Q along a curve C in space is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Unravel the meaning of this expression: If C has the parametric equations $x = x(t)$; $y = y(t)$; $z = z(t)$; $a \leq t \leq b$, then C is the graph of the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. So $\mathbf{F} \cdot d\mathbf{r} = \langle M, N, P \rangle \cdot \langle dx, dy, dz \rangle = M dx + N dy + P dz$. Hence we can write

$$W = \int_C [M dx + N dy + P dz].$$

Remark:

The formula $W = \int_C \mathbf{F} \cdot d\mathbf{r}$ is more useful when \mathbf{F} is a conservative vector field because you could then find a function f so that $\nabla f = \mathbf{F}$ and apply the Fundamental Theorem for Line Integrals (see section 14.3). Otherwise, use the formula $W = \int_C [M dx + N dy]$, or $W = \int_C [M dx + N dy + P dz]$.

Fundamental Theorem for Line Integrals: Section 14.3

Suppose C is a piecewise smooth curve given by the curvilinear motion $\mathbf{r} = \mathbf{r}(t)$; $\alpha \leq t \leq \beta$. Let $\mathbf{a} = \mathbf{r}(\alpha)$ denote the start of C , and let $\mathbf{b} = \mathbf{r}(\beta)$ denote the end of C . If f (a scalar function) is continuously differentiable on an open set containing C , then

$$\int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}).$$

Green's Theorem: Section 14.4

Let C be a piecewise, smooth, and simple curve that encloses a region S in the xy -plane. If $M(x, y)$ and $N(x, y)$ have continuous partial derivatives on S and on C , then

$$\oint_C [M dx + N dy] = \iint_S (N_x - M_y) dA.$$

Remarks:

- 1.) Green's Theorem equates a line integral with a double integral.
- 2.) The circle on the line integral denotes that we integrate over C in a counterclockwise direction.
- 3.) C must be a closed loop in this theorem.
- 4.) C is a plane curve. So you should have 2 components (not 3) when parameterizing C .
- 5.) C forms the boundary of a region S . The boundary of S is denoted ∂S . So Green's Theorem might read

$$\oint_{\partial S} [M dx + N dy] = \iint_S (N_x - M_y) dA.$$

Keep in mind that ∂S in this situation is a curve (not a surface).

Surface Integrals: Section 14.5

Let R be a region lying in the xy -plane. Let G be a surface given by $z = f(x, y)$ with (x, y) in R . If f has continuous first-order partial derivatives and $g(x, y, z) = g(x, y, f(x, y))$ is continuous on R , then

$$\iint_G g(x, y, z) dS = \iint_R g(x, y, f(x, y)) \sqrt{f_x^2 + f_y^2 + 1} dA.$$

Remark:

If $g(x, y, z) = 1$, then this formula will simply give you the surface area of G (see section 13.6).

Gauss's Divergence Theorem: Section 14.6

Let S be a 3-D solid enclosed by a smooth surface ∂S , and let \mathbf{n} be the unit normal vector of ∂S that points outward of S . If $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is a vector field such that M , N , and P have continuous first-order partial derivatives, then

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_S (M_x + N_y + P_z) \, dV.$$

Remarks:

- 1.) Gauss's Divergence Theorem equates a surface integral with a triple integral.
- 2.) Here, ∂S is a surface (not a curve).
- 3.) $\|\mathbf{n}\| = 1$.
- 4.) From Section 14.1, we know that $\operatorname{div}\mathbf{F} = M_x + N_y + P_z$. So Gauss's Divergence Theorem might read

$$\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_S \operatorname{div}\mathbf{F} \, dV.$$

Stokes's Theorem: Section 14.7

Let S be an orientable surface in space, and let \mathbf{n} be an upward pointing unit normal vector emanating from S . Suppose the boundary, ∂S is a piecewise smooth, simple curve that is oriented consistently with \mathbf{n} . If $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is a vector field such that M , N , and P have continuous first-order partial derivatives, then

$$\iint_S (\operatorname{curl}\mathbf{F}) \cdot \mathbf{n} \, dS = \oint_{\partial S} [M \, dx + N \, dy + P \, dz].$$

Remarks:

- 1.) Stokes's Theorem equates a surface integral with a line integral.
- 2.) Here, ∂S is a curve (not a surface). Furthermore, it is a curve in space, so you should have 3 components (not 2) when parameterizing ∂S .
- 3.) Once again, the circle on the integral denotes that we integrate over ∂S in a counterclockwise direction.
- 4.) $\|\mathbf{n}\| = 1$.
- 5.) From Section 14.1, we know that $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$. So Stokes's Theorem might read

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_{\partial S} [M \, dx + N \, dy + P \, dz].$$