Integral Formulas from Chapter 17:

Line Integrals:

1.) If C is a piecewise smooth curve in the plane, and it has the parametric equations $x = x(t)$; $y = y(t)$; $a \leq t \leq b$, then

$$\int_C f(x,y)\ ds = \int_a^b f\left(x(t), y(t)\right) \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\ dt.$$

2.) If C is a piecewise smooth curve in space, and it has the parametric equations $x = x(t)$; $y = y(t)$; $z = z(t)$; $a \leq t \leq b$, then

$$\int_C f(x,y,z)\ ds = \int_a^b f\left(x(t), y(t), z(t)\right) \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2 + \left[z'(t)\right]^2}\ dt.$$

Work Integrals:

1.) Let $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ be a force field acting on a point $Q = (x,y)$ in the plane. The amount of work required to move Q along a curve C in the plane is

$$W = \int_C \vec{F} \cdot d\vec{r}.$$

Unravel the meaning of this expression: If C has the parametric equations $x = x(t)$; $y = y(t)$; $a \leq t \leq b$, then C is the graph of the vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$. So $\vec{F} \cdot d\vec{r} = \langle M, N\rangle \cdot \langle dx, dy\rangle = M\ dx + N\ dy$. Hence we can write

$$W = \int_C [M\ dx + N\ dy].$$

2.) Let $\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ be a force field acting on a point $Q = (x,y,z)$ in space. The amount of work required to move Q along a curve C in space is

$$W = \int_C \vec{F} \cdot d\vec{r}.$$

Unravel the meaning of this expression: If C has the parametric equations $x = x(t)$; $y = y(t)$; $z = z(t)$; $a \leq t \leq b$, then C is the graph of the vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$. So $\vec{F} \cdot d\vec{r} = \langle M, N, P\rangle \cdot \langle dx, dy, dz\rangle = M\ dx + N\ dy + P\ dz$. Hence we can write

$$W = \int_C [M\ dx + N\ dy + P\ dz].$$

Remark:

The formula $W = \int_C \vec{F} \cdot d\vec{r}$. comes in handy if $\vec{F}$ is a conservative vector field because you could then apply the Fundamental Theorem for Line Integrals. Otherwise, just use $W = \int_C [M\ dx + N\ dy]$, or $W = \int_C [M\ dx + N\ dy + P\ dz]$.

Fundamental Theorem for Line Integrals:

If C is a piecewise smooth curve given parametrically by $\vec{r} = \vec{r}(t)$; $\alpha \le t \le \beta$, then C starts at $\vec{a} = \vec{r}(\alpha)$ and ends at $\vec{b} = \vec{r}(\beta)$. If f (a scalar function) is continuously differentiable on an open set containing C, then

$$\int_C \nabla f(\vec{r}) \cdot d\vec{r} = f(\vec{b}) - f(\vec{a}).$$

Green's Theorem:

Let C be a piecewise, smooth, and simple curve that encloses a region S in the xy-plane. If $M(x, y)$ and $N(x, y)$ have continuous partial derivatives on S and on C, then

$$\oint_C [M\ dx + N\ dy] = \iint_S (N_x - M_y)\ dA.$$

<u>Remarks:</u>

1.) Green's Theorem equates a line integral with a double integral.

2.) The circle on the line integral denotes that we integrate over C in a counterclockwise direction.

3.) C must be a closed loop in this theorem.

4.) C is a plane curve. So you should have 2 components (not 3) when parameterizing C.

5.) C forms the boundary of a region S. The boundary of S is denoted ∂S. So Green's Theorem might read

$$\oint_{\partial S} [M\ dx + N\ dy] = \iint_S (N_x - M_y)\ dA.$$

Keep in mind that ∂S in this situation is a curve (not a surface).

Surface Integrals:

Let R be a region lying in the xy-plane. Let G be a surface given by $z = f(x, y)$ with (x, y) in R. If f has continuous first-order partial derivatives and $g(x, y, z) = g(x, y, f(x, y))$ is continuous on R, then

$$\iint_G g(x, y, z)\ dS = \iint_R g(x, y, f(x, y)) \sqrt{f_x^2 + f_y^2 + 1}\ dA.$$

<u>Remark:</u>

If $g(x, y, z) = 1$, then this formula will simply give you the surface area of G (See section 16.6).

Gauss's Divergence Theorem:
Let S be a 3-D solid enclosed by a smooth surface ∂S, and let $\vec{n}$ be the unit normal vector of ∂S that points outward of S. If $\vec{F}(x, y, z) = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}$ is a vector field such that M, N, and P have continuous first-order partial derivatives, then

$$\iint_{\partial S} \vec{F} \cdot \vec{n}\ dS = \iiint_S (M_x + N_y + P_z)\ dV.$$

Remarks:
1.) Gauss's Divergence Theorem equates a surface integral with a triple integral.
2.) Here, ∂S is a surface (not a curve).
3.) $|\vec{n}| = 1$.
4.) From Section 17.1, we know that $\text{div}\vec{F} = M_x + N_y + P_z$. So Gauss's Divergence Theorem might read

$$\iint_{\partial S} \vec{F} \cdot \vec{n}\ dS = \iiint_S \text{div}\vec{F}\ dV.$$

Stokes's Theorem:
Let S be an orientable surface in space, and let $\vec{n}$ be an upward pointing unit normal vector emanating from S. Suppose the boundary, ∂S is a piecewise smooth, simple curve that is oriented consistently with $\vec{n}$. If $\vec{F}(x, y, z) = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}$ is a vector field such that M, N, and P have continuous first-order partial derivatives, then

$$\iint_S \left(\text{curl}\vec{F}\right) \cdot \vec{n}\ dS = \oint_{\partial S} [M\ dx + N\ dy + P\ dz].$$

Remarks:
1.) Stokes's Theorem equates a surface integral with a line integral.
2.) Here, ∂S is a curve (not a surface). Furthermore, it is a curve in space, so you should have 3 components (not 2) when parameterizing ∂S.
3.) Once again, the circle on the integral denotes that we integrate over ∂S in a counterclockwise direction.
4.) $|\vec{n}| = 1$.
5.) From Section 17.1, we know that $\text{curl}\vec{F} = \nabla \times \vec{F}$. So Stokes's Theorem might read

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS = \oint_{\partial S} [M\ dx + N\ dy + P\ dz].$$