

1. Let  $\vec{u} = \langle 1, 4, 2 \rangle$ ,  $\vec{v} = \langle -2, 3, 1 \rangle$ , and  $\vec{w} = \langle 4, -1, -1 \rangle$ . Perform the indicated operations.

(a) Calculate  $(2\vec{u} + 3\vec{v}) \cdot \vec{w}$ .

$$\begin{aligned} (2\vec{u} + 3\vec{v}) \cdot \vec{w} &= (\langle 2, 8, 4 \rangle + \langle -6, 9, 3 \rangle) \cdot \langle 4, -1, -1 \rangle = \langle -4, 17, 7 \rangle \cdot \langle 4, -1, -1 \rangle \\ &= -4(4) + 17(-1) + 7(-1) = -16 - 17 - 7 = -40. \end{aligned}$$

(b) Compute the angle (in degrees) between  $\vec{u}$  and  $\vec{v}$ .

$$\begin{aligned} \text{Use the formula } \vec{u} \cdot \vec{v} &= |\vec{u}| |\vec{v}| \cos \theta. \quad \vec{u} \cdot \vec{v} = 1(-2) + 4(3) + 2(1) = -2 + 12 + 2 = 12. \\ |\vec{u}| &= \sqrt{1^2 + 4^2 + 2^2} = \sqrt{1 + 16 + 4} = \sqrt{21}. \quad |\vec{v}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}. \end{aligned}$$

$$\text{Thus, } 12 = \sqrt{21}\sqrt{14} \cos \theta \Rightarrow 12 = 7\sqrt{6} \cos \theta \Rightarrow \frac{2\sqrt{6}}{7} = \cos \theta. \text{ Therefore, } \theta \approx 45.5847^\circ.$$

(c) Calculate  $\vec{v} \times \vec{w}$ .

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 2 \\ -2 & 3 & 1 \end{vmatrix} = (-3 + 1)\vec{i} - (2 - 4)\vec{j} + (2 - 12)\vec{k} = -2\vec{i} + 2\vec{j} - 10\vec{k} = \langle -2, 2, -10 \rangle.$$

(d) Find the volume of the parallelepiped determined by  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$ .

$$\begin{aligned} \text{We can use 2 methods. } |\vec{u} \cdot (\vec{v} \times \vec{w})| &= |\langle 1, 4, 2 \rangle \cdot \langle -2, 2, -10 \rangle| \text{ from part (b). This equals} \\ |1(-2) + 4(2) + 2(-10)| &= |-2 + 8 - 20| = |-14| = 14. \end{aligned}$$

$$\text{Another method we can use is to find } \left| \begin{vmatrix} 1 & 4 & 2 \\ -2 & 3 & 1 \\ 4 & -1 & -1 \end{vmatrix} \right|, \text{ which is}$$

$$|(-3 - 1)(1) - (2 - 4)(4) + (2 - 12)(2)| = |-4(1) + 2(4) - 10(2)| = |-4 + 8 - 20| = |-14| = 14.$$

2. Let  $\vec{u} = \langle 3, 4 \rangle$  and  $\vec{v} = \langle 5, 12 \rangle$ . We can write  $\vec{u} = \vec{w} + \vec{n}$ , where  $\vec{w}$  is parallel to  $\vec{v}$  and  $\vec{n}$  is perpendicular to  $\vec{v}$ . Find  $\vec{w}$  and  $\vec{n}$ .

$$\vec{w} = \text{pr}_{\vec{v}}\vec{u} = \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}. \quad \vec{u} \cdot \vec{v} = 3(5) + 12(4) = 15 + 48 = 63. \quad |\vec{v}|^2 = 5^2 + 12^2 = 25 + 144 = 169.$$

$$\text{Thus, } \vec{w} = \frac{63}{169} \langle 5, 12 \rangle = \left\langle \frac{315}{169}, \frac{756}{169} \right\rangle.$$

$$\text{Then } \vec{n} = \vec{u} - \vec{w} = \langle 3, 4 \rangle - \left\langle \frac{315}{169}, \frac{756}{169} \right\rangle = \left\langle \frac{517}{169}, \frac{676}{169} \right\rangle - \left\langle \frac{315}{169}, \frac{756}{169} \right\rangle = \left\langle \frac{192}{169}, -\frac{80}{169} \right\rangle.$$

3. Find the equation of the plane that passes through the points  $P = (1, 4, 6)$ ,  $Q = (-2, 5, -1)$ , and  $R = (1, -1, 1)$ .

If  $\vec{u}$  is the vector from Q to P, then  $\vec{u} = \langle (1+2), (4-5), (6+1) \rangle = \langle 3, -1, 7 \rangle$ . Similarly, if  $\vec{v}$  is the vector from Q to R, then  $\vec{v} = \langle (1+2), (-1-5), (1+1) \rangle = \langle 3, -6, 2 \rangle$ . To compute a normal to the plane, we compute  $\vec{u} \times \vec{v}$ .

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 7 \\ 3 & -6 & 2 \end{vmatrix} = (-2 + 42)\vec{i} - (6 - 21)\vec{j} + (-18 + 3)\vec{k} = 40\vec{i} = 15\vec{j} - 15\vec{k} = \langle 40, 15, -15 \rangle.$$

This vector has the same direction as  $\langle 8, 3, -3 \rangle$ . So the equation of the plane has the form

$8x + 3y - 3z = D$ . Using the point P, we get  $8(1) + 3(4) - 3(6) = D \Rightarrow D = 8 + 12 - 18 = 2$ . Therefore, the equation of the plane is  $8x + 3y - 3z = 2$ .

4. A particle P travels in the plane, and its vector position at time  $t$  is given by  $\vec{r}(t) = (1-t)\vec{i} + \sqrt{1-t^2}\vec{j}$ .

- (a) Find the Cartesian equation for the path of the particle.

From the vector equation, the parametric equations are given by  $x = 1 - t$  and  $y = \sqrt{1 - t^2}$ . Solve the first equation for  $t$  and get  $t = 1 - x$ . Plug into the second equation to get

$$y = \sqrt{1 - (1-x)^2} = \sqrt{1 - (1 - 2x + x^2)} = \sqrt{2x - x^2}. \quad \text{Hence, the path of the particle is } y^2 = 2x - x^2 \text{ in Cartesian coordinates.}$$

- (b) Assuming that P starts moving at time  $t = 0$ , how far will it have traveled at  $t = 1/2$  seconds?

$$L = \int_0^{1/2} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt. \quad x'(t) = -1, \text{ and so } [x'(t)]^2 = 1.$$

$$y'(t) = \frac{1}{2}(1-t^2)^{-1/2}(-2t) = -\frac{t}{\sqrt{1-t^2}}, \text{ and so } [y'(t)]^2 = \frac{t^2}{1-t^2}.$$

$$\text{Hence, } [x'(t)]^2 + [y'(t)]^2 = 1 + \frac{t^2}{1-t^2} = \frac{1-t^2}{1-t^2} + \frac{t^2}{1-t^2} = \frac{1}{1-t^2}. \quad \text{Therefore,}$$

$$L = \int_0^{1/2} \sqrt{\frac{1}{1-t^2}} dt = \int_0^{1/2} \frac{1}{\sqrt{1-t^2}} dt = [\sin^{-1} t]_0^{1/2} = \sin^{-1}(1/2) - \sin^{-1}(0) = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

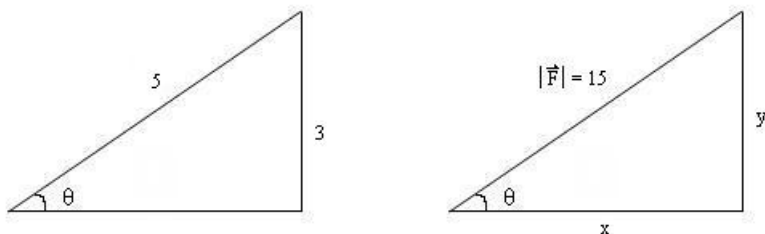
5. A small kid who goes by the nickname Triple A is going sledding. It is a quarter-mile (1320 feet) walk from his house to get to the sledding hill. He exerts 15 pounds of force by pulling the sled. The sled is tied to a 5 foot rope, and the hand that he pulls the rope with is 3 feet above the ground. How much work does Triple A do in pulling the sled? Assume that the ground is flat from his house to the sledding hill, and don't worry about friction.

To find the work, we need to find force,  $\vec{F}$ , and direction,  $\vec{D}$ . Since the rope is 5 feet long, and his hand is 3 feet off of the ground, we can compute the angle,  $\theta$ , between the ground and the rope. Using trigonometry,  $\sin \theta = \frac{3}{5}$ , so therefore,  $\theta \approx 36.8699^\circ$  (see figure on the left). Let  $x$  be the horizontal component of  $\vec{F}$ ,

and  $y$  be the vertical component of  $\vec{F}$ . Using trigonometry,  $\cos 36.8699 = \frac{x}{15} \Rightarrow x = 15 \cos 36.8699 = 12$ .

Similarly,  $\sin 36.8699 = \frac{y}{15} \Rightarrow y = 15 \sin 36.8699 = 9$  (see figure on the right). Therefore,  $\vec{F} = 12\vec{i} + 9\vec{j}$ .

Finally, since Triple A walks 1320 feet and the ground is flat,  $\vec{D} = 1320\vec{i}$ . Therefore, the amount of work done is  $\vec{F} \cdot \vec{D} = 12(1320) + 9(0) = 15,840$  foot-pounds.



Remark: An easier way to do this is to use the 3-4-5 Pythagorean triple. The left figure is 3-4-5, and the right figure is a similar triangle magnified by a factor of 3; a 9-12-15 triangle.

6. Find the parametric equations of the line of intersection between the planes  $x - 2y + 4z = 14$  and  $-x + 2y - 5z = -30$ .

First, we find the direction of the line. It will be parallel to both planes. But this means that it is perpendicular to both normal vectors pointing from the planes. So we will let  $\vec{n}_1$  be the normal pointing out of  $x - 2y + 4z = 14$  and  $\vec{n}_2$  be the normal pointing out of  $-x + 2y - 5z = -30$ . The direction vector  $\vec{v}$  will be the cross product of  $\vec{n}_1$  and  $\vec{n}_2$ . So  $\vec{n}_1 = \langle 1, -2, 4 \rangle$  and  $\vec{n}_2 = \langle -1, 2, -5 \rangle$ . So

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 4 \\ -1 & 2 & -5 \end{vmatrix} = (10 - 8)\vec{i} - (-5 + 4)\vec{j} + (2 - 2)\vec{k} = \langle 2, 1, 0 \rangle.$$

Next, we find a point the line goes through. I will choose to let  $x = 0$ . Then I get the equations  $-2y + 4z = 14$  and  $2y - 5z = -30$ . Adding the two together gives us  $-z = -16 \Rightarrow z = 16$ . Then plug into one of the two (I'll plug into the first equation) and get  $-2y + 64 = 14 \Rightarrow -2y = -50 \Rightarrow y = 25$ . Therefore, the line goes through the point  $(0, 25, 16)$ . So the parametric equations for the line are  $x = 2t, y = 25 + t, z = 16$ .

Note: If you use  $y = 0$ , you get  $x = 2t - 50, y = t, z = 16$ . Since  $z$  has to be 16, you'll get no solution for  $x$  and  $y$  if  $z = 0$ .

7. For each of the following, write the equations in a standard form, and identify the surface in space they describe. You don't need to graph them.

(a)  $441x^2 + 144y^2 + 784z^2 - 7056 = 0$ .

First, add 7056 to both sides and get  $441x^2 + 144y^2 + 784z^2 = 7056$ . Divide both sides by 7056 to obtain  $\frac{x^2}{16} + \frac{y^2}{49} + \frac{z^2}{9} = 1$ . This is the equation of an ellipsoid with  $x$ -radius = 4,  $y$ -radius = 7, and  $z$ -radius = 3.

(b)  $7x - 4y^2 + 3z^2 = 0$

Notice that the  $x$  is not raised to the second power. So we'll isolate it. Adding  $4y^2 - 3z^2$  to both sides yields  $7x = 4y^2 - 3z^2$ . Then divide both sides by 7 to get  $x = \frac{4}{7}y^2 - \frac{3}{7}z^2$ . This is the equation of a hyperbolic paraboloid.

(c)  $x^2 + z^2 - 8x + 4z + 13 = 0$

Here, we have two variables not raised to the second power. This is a tipoff to apply the method of completing the square. Grouping  $x$  and  $z$  terms together on the left side and subtracting 13 from both sides yields  $(x^2 - 8x) + (z^2 + 4z) = -13$ . Now complete the square as follows:

$(x^2 - 8x + 16) + (z^2 + 4z + 4) = -13 + 16 + 4 \Rightarrow (x - 4)^2 + (z + 2)^2 = 7$ . This is the equation of a right circular cylinder whose central axis is the line  $x = 4$ ,  $z = -2$  and has radius  $\sqrt{7}$ .

(d)  $2x^2 + 2y^2 + 2z^2 - 12x - 6y + 28z + 81 = 0$

Here we have three variables not raised to the second power. So once again, we resort to completing the square. But first, we must divide both sides by 2 and get,  $x^2 + y^2 + z^2 - 6x - 3y + 14z + \frac{81}{2} = 0$ .

Grouping  $x$ ,  $y$ , and  $z$  terms together on the left side and subtracting  $\frac{81}{2}$  from both sides yields

$(x^2 - 6x) + (y^2 - 3y) + (z^2 + 14z) = -\frac{81}{2}$ . Now complete the square to get

$$(x^2 - 6x + 9) + \left(y^2 - 3y + \frac{9}{4}\right) + (z^2 + 14z + 49) = -\frac{81}{2} + 9 + \frac{9}{4} + 49$$

$$\Rightarrow (x - 3)^2 + \left(y - \frac{3}{2}\right)^2 + (z + 7)^2 = \frac{-162 + 36 + 9 + 196}{4}$$

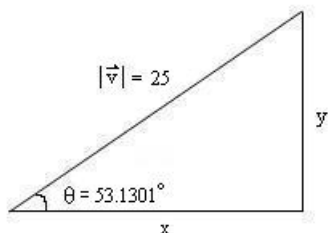
$$\Rightarrow (x - 3)^2 + \left(y - \frac{3}{2}\right)^2 + (z + 7)^2 = \frac{79}{4}$$

This is the equation of a sphere centered at  $\left(3, \frac{3}{2}, -7\right)$  of radius  $\frac{\sqrt{79}}{2}$ .

8. A cannon sits atop a cliff that is 176 feet tall. It is aimed at an angle of  $53.1301^\circ$  above the horizontal, and it shoots a projectile at the speed of 25 feet per second.

- (a) Find the initial velocity vector of the projectile.

We'll denote the initial velocity vector  $\vec{v}(0)$ . We are given that  $|\vec{v}(0)| = 25$ , and the angle between  $\vec{v}(0)$  and the ground is  $53.1301^\circ$ . Let  $x$  be the horizontal component of the velocity and  $y$  be the vertical component of the velocity.



Then using trigonometry,  $\cos 53.1301 = \frac{x}{25} \Rightarrow x = 25 \cos 53.1301 = 15$ . Similarly,  $\sin 53.1301 = \frac{y}{25} \Rightarrow y = 25 \sin 53.1301 = 20$ . Therefore,  $\vec{v}(0) = 15\vec{i} + 20\vec{j}$ .

- (b) Using part(a), the fact that  $\vec{a}(t) = -32\vec{j}$ , and assuming  $\vec{r}(0) = 176\vec{j}$ , find the velocity,  $\vec{v}(t)$ , and the position,  $\vec{r}(t)$ , of the projectile for any time  $t$ .

Taking the antiderivative of  $\vec{a}(t)$  yields  $\vec{v}(t) = C_1\vec{i} + (-32t + C_2)\vec{j}$ . At  $t = 0$ ,  $v(0) = C_1\vec{i} + C_2\vec{j}$ . But from part (a), this should be  $15\vec{i} + 20\vec{j}$ . Hence,  $C_1 = 15$ ,  $C_2 = 20$ , and thus  $\vec{v}(t) = 15\vec{i} + (-32t + 20)\vec{j}$ . Now, we antidifferentiate again to get  $\vec{r}(t) = (15t + C_3)\vec{i} + (-16t^2 + 20t + C_4)\vec{j}$ . At  $t = 0$ ,  $\vec{r}(0) = C_3\vec{i} + C_4\vec{j}$ . But this should be  $176\vec{j}$ . Therefore,  $C_3 = 0$ ,  $C_4 = 176$ , and thus  $\vec{r}(t) = 15t\vec{i} + (-16t^2 + 20t + 176)\vec{j}$ .

- (c) At what time does the projectile hit the ground?

This happens when the vertical component of the position is zero. That is, when  $-16t^2 + 20t + 176 = 0 \Rightarrow 4t^2 - 5t - 44 = 0$ . You can use the quadratic formula to find  $t$ , but luckily  $4t^2 - 5t - 44$  factors into  $(4t + 11)(t - 4)$ . Thus,  $4t + 11 = 0$  or  $t - 4 = 0$ , which implies  $t = -\frac{11}{4}$  or  $t = 4$ . Since  $t = -\frac{11}{4}$  is preposterous, we must have  $t = 4$ . So the projectile hits the ground 4 seconds after it is shot.

- (d) With what speed does the projectile come smashing into the basin below the cliff?

First, we need to find the velocity at  $t = 4$ .  $\vec{v}(4) = 15\vec{i} + (-32(4) + 20)\vec{j} = 15\vec{i} - 108\vec{j}$ . Then, the speed is  $|\vec{v}(4)| = \sqrt{15^2 + 108^2} = \sqrt{225 + 11,664} = \sqrt{11,889} \approx 109.037$  feet per second.

9. Let  $\vec{r}(t) = t\vec{i} + t^2\vec{j} + \frac{2}{3}t^3\vec{k}$ .

(a) Find  $\vec{T}(1)$ .

$$\vec{v}(t) = \vec{i} + 2t\vec{j} + 2t^2\vec{k}. \text{ Then } \vec{v}(1) = \vec{i} + 2\vec{j} + 2\vec{k}. \text{ Therefore,}$$

$$\vec{T}(1) = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{\sqrt{1+4+4}} = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{\sqrt{9}} = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle.$$

(b) Find  $\vec{N}(t)$ .

$$\vec{a}(t) = 2\vec{j} + 4t\vec{k}. \text{ Then } \vec{a}(1) = 2\vec{j} + 4\vec{k}. \text{ Therefore,}$$

$$a_T(1) = \vec{T}(1) \cdot \vec{a}(1) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle \cdot \langle 0, 2, 4 \rangle = 0 + \frac{4}{3} + \frac{8}{3} = \frac{12}{3} = 4.$$

$$a_N(1) = |\vec{T}(1) \times \vec{a}(1)| = \left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1/3 & 2/3 & 2/3 \\ 0 & 2 & 4 \end{vmatrix} \right| = \left| \left\langle \left(\frac{8}{3} - \frac{4}{3}\right), -\left(\frac{4}{3} - 0\right), \left(\frac{2}{3} - 0\right) \right\rangle \right|$$

$$= \left| \left\langle \frac{4}{3}, -\frac{4}{3}, \frac{2}{3} \right\rangle \right| = \sqrt{\frac{16}{9} + \frac{16}{9} + \frac{4}{9}} = \sqrt{\frac{36}{9}} = \sqrt{4} = 2.$$

Then, since  $\vec{a}(1) = a_T(1)\vec{T}(1) + a_N(1)\vec{N}(1)$ , we can compute  $\vec{N}(1)$  as follows:

$$\vec{N}(1) = \frac{\vec{a}(1) - a_T(1)\vec{T}(1)}{a_N(1)} = \frac{\langle 0, 2, 4 \rangle - 4\langle 1/3, 2/3, 2/3 \rangle}{2} = \frac{1}{2} \left( \langle 0, 2, 4 \rangle - \left\langle \frac{4}{3}, \frac{8}{3}, \frac{8}{3} \right\rangle \right)$$

$$= \frac{1}{2} \left( \left\langle 0, \frac{6}{3}, \frac{12}{3} \right\rangle - \left\langle \frac{4}{3}, \frac{8}{3}, \frac{8}{3} \right\rangle \right) = \frac{1}{2} \left\langle -\frac{4}{3}, -\frac{2}{3}, \frac{4}{3} \right\rangle = \left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle.$$

(c) Find the curvature,  $\kappa$  at  $t = 1$ .

$$\kappa(1) = \frac{|\vec{v}(1) \times \vec{a}(1)|}{|\vec{v}(1)|^3}. \text{ But } a_N(1) = \frac{|\vec{v}(1) \times \vec{a}(1)|}{|\vec{v}(1)|}. \text{ Thus, we can use our calculation for } a_N(1) \text{ in}$$

$$\text{part (b) to get } \kappa = \frac{a_N(1)}{|\vec{v}(1)|^2} = \frac{2}{1 + 2^2 + 2^2} = \frac{2}{9}.$$

(d) Find the binormal  $\vec{B}(1)$ .

$$\vec{B}(1) = \vec{T}(1) \times \vec{N}(1) = \left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \end{vmatrix} \right| = \left\langle \left(\frac{4}{9} + \frac{2}{9}\right), -\left(\frac{2}{9} + \frac{4}{9}\right), \left(\frac{-1}{9} + \frac{4}{9}\right) \right\rangle$$

$$= \left\langle \frac{6}{9}, -\frac{6}{9}, \frac{3}{9} \right\rangle = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle.$$