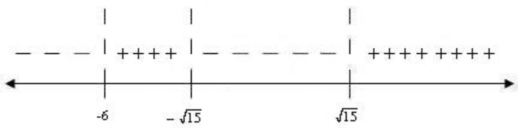
Allen – Math 165 E1 Exam #3 Name: ____________________

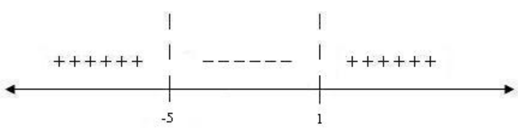
1. Consider the function $f(x) = \dfrac{x^4}{4} + 2x^3 - \dfrac{15x^2}{2} - 90x + \dfrac{225}{4}$.

   (a) Determine the interval(s) on which $f(x)$ is increasing and decreasing (Hint: Use factoring by grouping to help you find critical points).
   Since $f(x)$ is a polynomial, there are no singular points, and there are no endpoints. So the only critical points will be stationary points.
   $f'(x) = x^3 + 6x^2 - 15x - 90 = x^2(x+6) - 15(x+6) = (x^2 - 15)(x+6)$.
   So $f'(x) = 0$ implies that $x = \pm\sqrt{15}$ or $x = -6$. So these are the three critical points. Choosing test points $x = -7$, $x = -4$, $x = 0$ and $x = 4$, we find that $f'(-7) = -34 < 0$, $f'(-4) = 2 > 0$, $f'(0) = -90 < 0$, and $f'(10) = 10 > 0$.



   Therefore, f is increasing on $\left[-6, -\sqrt{15}\right] \cup \left[\sqrt{15}, \infty\right)$, and decreasing on $(-\infty, -6] \cup \left[-\sqrt{15}, \sqrt{15}\right]$.

   (b) Determine the interval(s) on which $f(x)$ is concave up and concave down.
   $f''(x) = 3x^2 + 12x - 15 = 3(x^2 + 4x - 5) = 3(x-1)(x+5)$. So $f''(x) = 0$ implies that $x = 1$ or $x = -5$. Doing a number line analysis on f'', we choose test points $x = -6$, $x = 0$, and $x = 2$ to find that $f''(-6) = 21 > 0$, $f''(0) = -15 < 0$, and $f''(2) = 21 > 0$.



   Hence f is concave up on $(-\infty, -5) \cup (1, \infty)$ and concave down on $(-5, 1)$.

   (c) Identify all x-values for which there are local maxima, local minima, and inflection points. You don't need to find the y-coordinates of these points.
   Using the second derivative test, notice that $f''(-6) > 0$, $f''(-\sqrt{15}) < 0$, and $f''(\sqrt{15}) > 0$. So f has a local maximum at $x = -\sqrt{15}$, and local minima at $x = -6$, and $x = \sqrt{15}$. Finally, f has inflection points at $x = -5$ and $x = 1$, since f changes concavity at these points (according to part b) and f is continuous at $x = -5$ and $x = 1$.

2. Evaluate the following:

(a) $\displaystyle\int \left(5x^{17} + x^{11} - 5x^2 + \frac{4}{x^3} - \frac{1}{x^{13}}\right)\, dx.$

$\displaystyle\int \left(5x^{17} + x^{11} - 5x^2 + 4x^{-3} - x^{-13}\right)\, dx = \frac{5x^{18}}{18} + \frac{x^{12}}{12} - \frac{5x^3}{3} + \frac{4x^{-2}}{-2} - \frac{x^{-12}}{-12} + C$

$\displaystyle= \frac{5x^{18}}{18} + \frac{x^{12}}{12} - \frac{5x^3}{3} - \frac{2}{x^2} + \frac{1}{12x^{12}} + C.$

(b) $\displaystyle\int \tan^{11} x \cdot \sec^2 x\, dx$ (Hint: Recall $D_x[\tan x] = \sec^2 x$).

Apply a u-substitution. Let $u = \tan x$. Then $du = \sec^2 x\, dx$. Thus we have

$\displaystyle\int (\tan x)^{11} \cdot \sec^2 x\, dx = \int u^{11}\, du = \frac{u^{12}}{12} + C = \frac{\tan^{12} x}{12} + C.$

3. Find the function $s(t)$ that satisfies the following:

$\displaystyle\frac{ds}{dt} = 12s^3(3t^2 - t);\ s(1) = 1$

Here, we separate variables. Multiply both sides by dt and divide by $12s^3$ to get $\displaystyle\frac{ds}{12s^3} = (3t^2 - t)\, dt \Rightarrow$ $\displaystyle\frac{1}{12}s^{-3}\, ds = (3t^2 - t)\, dt$. Integrate both sides to get $\displaystyle\frac{1}{12}\int s^{-3}\, ds = \int (3t^2 - t)\, dt$

$\displaystyle\Rightarrow \frac{1}{12}\left[\frac{s^{-2}}{-2}\right] = t^3 - \frac{t^2}{2} + C_1 \Rightarrow -\frac{1}{24s^2} = t^3 - \frac{t^2}{2} + C_1 \Rightarrow \frac{1}{s^2} = -24t^3 + 12t^2 + C_2$

$\displaystyle\Rightarrow s^2 = \frac{1}{-24t^3 + 12t^2 + C_2} \Rightarrow s^2 = \frac{-1}{24t^3 - 12t^2 + C} \Rightarrow s = \sqrt{\frac{-1}{24t^3 - 12t^2 + C}}$. Next, we find C so that $s(1) = 1$. This means that C must satisfy $\displaystyle 1 = \sqrt{\frac{-1}{24(1)^3 - 12(1)^2 + C}} \Rightarrow 1 = \frac{-1}{24 - 12 + C}$

$\displaystyle\Rightarrow 1 = -24 + 12 + C \Rightarrow C = 13$. Therefore, $\displaystyle s(t) = \sqrt{\frac{-1}{24t^3 - 12t^2 + 13}}$.

4. Use Newton's method to find the solution to the equation $x\cos x = 1$ for $0 \le x \le 6$. The solution should be accurate to 9 decimal places.

Let $f(x) = x\cos x - 1$. This reduces the problem to finding the root of $f(x)$ on the interval $[0, 6]$. By inspecting the graph of $y = f(x)$, it looks like there is a root close to $x = 5$. So let $x_0 = 5$. Then for $n = 1, 2, \cdots$, $\displaystyle x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Here, $f'(x) = \cos x - x\sin x$. Using the handy-dandy calculator trick, we quickly get the following numbers:

n	x_n
1	4.917627497
2	4.917185944
3	4.917185925
4	4.917185925

So the solution is $x \approx 4.917185925$.

5. Consider the function $f(x) = x^3 + 5x^2 - 7x + 21$ on the interval $[-6, 2]$. Find each value of c in between -6 and 2 which satisfies the Mean Value Theorem.
According to the Mean Value Theorem, there is at least one number c in the interval $(-6, 2)$ such that $f'(c) = \frac{f(2) - f(-6)}{2 - (-6)}$. $f'(c) = 3c^2 + 10c - 7$. We also have $\frac{f(2) - f(-6)}{2 - (-6)} = \frac{(8 + 20 - 14 + 21) - (-216 + 180 + 42 + 21)}{8} = \frac{35 - 27}{8} = \frac{8}{8} = 1$. So we need to find any c that satisfies $3c^2 + 10c - 7 = 1$. Solving for c yields $3c^2 + 10c - 8 = 0 \Rightarrow (3c - 2)(c + 4) = 0$. So $c = \frac{2}{3}$ and $c = -4$ satisfy this equation. Furthermore, both values lie in the interval $(-6, 2)$. Therefore, $c = \frac{2}{3}$ and $c = -4$ satisfy the Mean Value Theorem for f.

6. Triple A is standing at the edge of a cliff that rises 768 feet above a basin. He is about to wreck some poor student's calculator by throwing it up and over the edge. The dirty rotten punk throws the calculator up in the air at an initial velocity of 32 feet per second. Assume that the acceleration due to gravity is $-32\frac{\text{ft}}{\text{sec}^2}$, and let the elevation of the basin be 0 feet.

   (a) Find the velocity of the calculator $v(t)$ for a given time $t > 0$.
   The velocity is the antiderivative of acceleration. So $v(t) = \int -32\ dt = -32t + C$. Since the initial velocity is 32 feet per second, $v(0) = 32$, and so $C = 32$. Therefore, velocity of the calculator at any given time $t > 0$ is $v(t) = -32t + 32$.

   (b) Find the position of the calculator $s(t)$ for a given time $t > 0$.
   The position is the antiderivative of velocity. So $s(t) = \int (-32t + 32)\ dt = -16t^2 + 32t + C$. Since the initial position is 768 feet, $s(0) = 768$, and so $C = 768$. Therefore, position of the calculator at any given time $t > 0$ is $s(t) = -16t^2 + 32t + 768$.

   (c) How fast does the calculator smash into the ground?
   First, we need to figure out what time the calculator hits the ground. This can be done by setting $s(t) = 0$. So we have $-16t^2 + 32t + 768 = 0 \Rightarrow t^2 - 2t - 48 = 0 \Rightarrow (t - 8)(t + 6) = 0$. Since $t = -6$ is nonsense, we conclude that the calculator hits the ground 8 seconds after it is thrown. Finally, we find how fast it hits the ground by plugging $t = 8$ into $v(t)$. Hence, the velocity is $v(8) = -32(8) + 32 = -224$ feet per second. Therefore, the calculator smashes into the ground at a speed of 224 feet per second.

7. Consider the function $f(x) = x^2$ on $[0, 2]$.

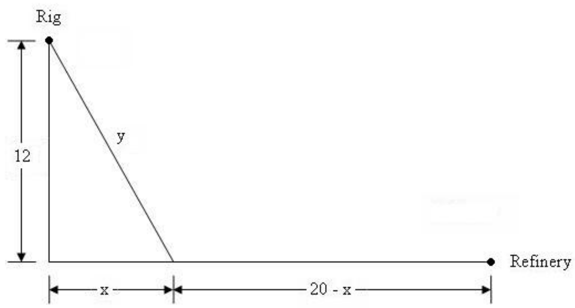
(a) Find the formula for the the area of the circumscribed polygon when the interval $[0, 2]$ is cut into n equally spaced subintervals (Suggestion: Use the right endpoint of the rectangles).

Since the circumscribed polygon corresponds to the right-hand rule, we calculate the Riemann sum $\sum_{i=1}^{n} f(x_i)\Delta x$. If we slice the interval $[0, 2]$ up into n equal pieces, each piece will have a length of $\Delta x = \frac{2}{n}$. Each x_i in the partition is the right edge of each rectangle. Observe that $x_1 = 2/n$, $x_2 = 4/n$, $x_3 = 6/n$, and so on. So $x_i = \frac{2i}{n}$ for each $i = 1, 2, \cdots, n$. Now we can set up our Riemann sum to get the formula for the area of the polygon, A_n, for any positive integer n.
$A_n = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right) \cdot \frac{2}{n} = \frac{2}{n} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 = \frac{2}{n} \sum_{i=1}^{n} \frac{4i^2}{n^2} = \frac{8}{n^3} \sum_{i=1}^{n} i^2$. Using one of the summation formulas, this implies that $A_n = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{4}{n^2}\left[\frac{(n+1)(2n+1)}{3}\right] = \frac{4(2n^2+3n+1)}{3}$.
Therefore, $A_n = \frac{8n^2 + 12n + 4}{3}$.

(b) Evaluate $\int_0^2 f(x)\,dx$ using part (a).

According to the definition of an indefinite integral, $\int_0^2 f(x)\,dx = \lim_{|P|\to 0} \sum_{i=1}^{n} f(\overline{x_i})\Delta x_i$ given any partition P of the interval $[0, 2]$ where $|P|$ is the widest subinterval of the partition. But the partition P in part(a) is n equally spaced pieces of length $\frac{2}{n}$. Furthermore, $\overline{x_i} = x_i = \frac{2i}{n}$ for $i = 1, 2, \cdots, n$ since we are using the right-hand sum. Finally, $|P| = 2/n$ and $|P| \to 0$ is exactly the same as $n \to \infty$. Therefore, $\int_0^2 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right) \cdot \frac{2}{n} = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{8n^2+12n+4}{3} = \frac{8}{3}$.

8. A drilling rig 12 miles offshore is to be connected by a pipe to a refinery onshore, 20 miles down the coast from the rig. If the underwater pipe costs \$50,000 per mile and the land-based pipe costs \$30,000 per mile, what values of x and y give the least expensive connection?



The total cost to build the pipeline in dollars will be $C = 50,000y + 30,000(20 - x)$. For convenience (so I don't have to keep writing a ton of zeros), I will let $\widehat{C} = 5y + 3(20 - x)$. Using the Pythagorean Theorem, $y = \sqrt{x^2 + 144}$. So $\widehat{C}(x) = 5\sqrt{x^2 + 144} + 3(20 - x)$. Then
$\widehat{C}'(x) = 5 \cdot \frac{1}{2}(x^2 + 144)^{-1/2} \cdot 2x - 3 = \frac{5x}{\sqrt{x^2 + 144}} - 3$. Now we solve $\widehat{C}'(x) = 0$ for x.
$\frac{5x}{\sqrt{x^2 + 144}} - 3 = 0 \Rightarrow \frac{5x}{\sqrt{x^2 + 144}} = 3 \Rightarrow 5x = 3\sqrt{x^2 + 144} \Rightarrow 25x^2 = 9(x^2 + 144) \Rightarrow 25x^2 = 9x^2 + 1296$
$\Rightarrow 16x^2 = 1296 \Rightarrow x^2 = 81 \Rightarrow x = \pm 9$.
Since $x = -9$ is nonsense, we have that the cost of the pipeline will be minimized if $x = 9$. Finally, $y = \sqrt{9^2 + 144} = \sqrt{81 + 144} = \sqrt{225} = 15$. Therefore, $x = 9$ and $y = 15$.

9. **Extra Credit:**

   (a) Evaluate $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$ without using calculus (Hint: Think geometry).

   By looking at the graph of the curve $y = \sqrt{4 - x^2}$, it is clear that the curve is just the upper half of a circle centered at 0 with radius 2. So the integral is the area of that semicircle, which is $\frac{1}{2}\pi(2)^2 = 2\pi$.

   (b) **Multiple Choice:** Who is Triple A?

   a.) An ambitious fellow who wants to build a crash test lab in his garage.
   b.) A punk who takes pleasure in smashing students calculators in strange ways.
   c.) Your instructor.
   d.) all of the above.

   (c) is the correct answer (my full name is Aaron Andrew Allen). For those of you who put (b) or (d), I am not a punk.