Stat 544 Final Exam

May 2, 2006

I have neither given nor received unauthorized assistance on this examination.

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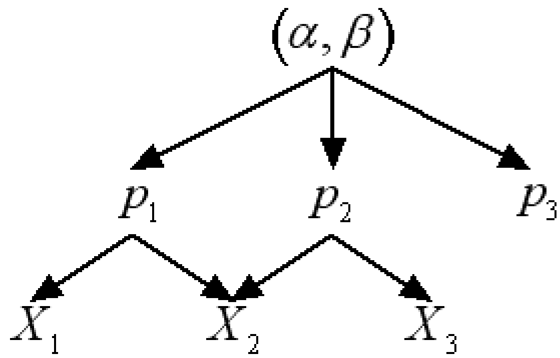
signature date

1. Below is a directed acyclic graph that represents a joint distribution for the variables $\alpha, \beta, p_1, p_2, p_3, X_1, X_2,$ and X_3.



Some information about this joint distribution is that:

α and β are independent, $\alpha \sim \text{Exp}(1)$, $\beta \sim \text{Exp}(1)$

conditioned on (α,β), p_1, p_2, p_3 are iid $\text{Beta}(\alpha,\beta)$

conditioned on p_1, p_2, p_3, the variables X_1, X_2, X_3 are independent, with

$$X_1 \sim \text{Bin}(100, p_1), X_2 \sim \text{Bin}(100, \min(p_1, p_2)), X_3 \sim \text{Bin}(100, p_2)$$

a) In general (for all distributions represented by this DAG) are X_1 and p_3 independent? Argue carefully that your answer is correct.

b) Are X_1 and p_3 conditionally independent given p_1? Argue carefully that your answer is correct.

c) Write out a joint "density" for these 8 random variables, $h(\alpha, \beta, p_1, p_2, p_3, x_1, x_2, x_3)$.

d) Suppose that $X_1 = 10, X_2 = 15,$ and $X_3 = 70$ and that one wishes to sample from the conditional distribution of the other variables given these values using a Gibbs sampler. Write out a function of p_1 proportional to the pdf on $(0,1)$ used in a "p_1 update" of the vector $(\alpha^*, \beta^*, p_1^*, p_2^*, p_3^*)$.

e) Finish the `WinBUGS` code below for implementing the Gibbs sampling alluded to in part d). (The `WinBUGS User Manual` says `step(e) is 1 if e>0; 0` otherwise.)

```
model {

  alpha ~

  beta ~

  for (           ) {

    p[i] ~

  }

  p<-(           )*step(           )+p[1]

  x1 ~ dbin (              )

  x2 ~ dbin (p,100)

  x3 ~ dbin (              )

  }

#data are next

list(                            )
```

2. Suppose that conditional on p_1 and p_2, X_1 and X_2 are independent with $X_i \sim \mathrm{Bin}(10, p_i)$. Two prior distributions for (p_1, p_2) are

Prior/Model 1: p_1, p_2 iid $\mathrm{U}(0,1)$

Prior/Model 2: $p_1 = p_2, p_1 \sim \mathrm{U}(0,1)$

Find the Bayes Factor for comparing these two models (likelihood plus prior) based on $X_1 = x_1$ and $X_2 = x_2$.

3. Two versions of an industrial process are run with the intention of comparing effectiveness. (There is an "old/#1" and a "new/#2" process.) Six different batches of raw material are used in the study. For a response variable y_{ij} = the yield of the *j*th run made using raw material batch i, we'll consider an analysis based on the model

$$y_{ij} = \mu_{\text{process}(ij)} + \beta_i + \varepsilon_{ij}$$

where $\text{process}(ij)$ takes the value either 1 or 2 depending upon which version of the process is used, μ_1 and μ_2 are mean yields for the two versions of the process, the β_i are iid $\text{N}\left(0,\sigma_\beta^2\right)$ independent of the ε_{ij} which are iid $\text{N}\left(0,\sigma^2\right)$, and the parameters of the model are $\mu_1, \mu_2, \sigma_\beta,$ and σ. Attached to this exam is a summary of a `WinBUGS` session run to analyze the total of $n = 25$ runs made in the study. Use it to answer the following questions.

a) Describe in standard mathematical (Stat 542) notation (NOT in `WinBUGS` terminology/notation) what prior distribution was used for $\left(\mu_1, \mu_2, \sigma_\beta^2, \sigma^2\right)$ in the analysis. (What was the form of $h\left(\mu_1, \mu_2, \sigma_\beta^2, \sigma^2\right)$?)

b) As it turns out, the 95% Bayes credible intervals for the parameters provided on the `WinBUGS` output are very much like the corresponding approximate 95% confidence intervals one obtains from an `R "lme"` mixed effects model analysis of the data. Why is this not surprising? Would you expect this to have been true if a substantially different prior had been used? Explain.

c) Is there a clear difference between the mean yields under the two different versions of the process? Explain.

d) What is a 95% credible interval for y_{new}, a yield for a future run of the production process made using a new batch of raw material and the "new/#2" version of the process?

4. A response, y, whose mean is known to increase with a covariate/predictor, x, is investigated in a study where (coded) values $x = 1, 2, 3, 4, 5, 6$ are used and there are 3 observations for each level of the predictor. Some summary statistics from that study are (in the obvious notation)

$$\bar{y}_{1.} = .51, \bar{y}_{2.} = -.15, \bar{y}_{3.} = .95, \bar{y}_{4.} = 5.88, \bar{y}_{5.} = 6.65, \bar{y}_{6.} = 7.59, \text{ and } \sqrt{MSE} = .99$$

Attached to this exam is summary of a Bayes analysis of the data based on the model

$$y_{xj} = \mu_x + \varepsilon_{xj} \qquad (*)$$

where $\mu_1 \le \mu_2 \le \mu_3 \le \mu_4 \le \mu_5 \le \mu_6$ are unknown means and the ε_{xj} are iid $\mathrm{N}(0, \sigma^2)$.

a) What is a 95% credible interval for $\mu_3 - \mu_2$ under the prior used in the Bayes analysis?

b) The prior used in the Bayes analysis for model (*) is improper. *Exactly* how would you modify the prior to create one that is proper, but should produce results close to those provided here? (Fully specify your alternative prior.)

c) One alternative to the model (*) with the order restriction on the 6 means is a model

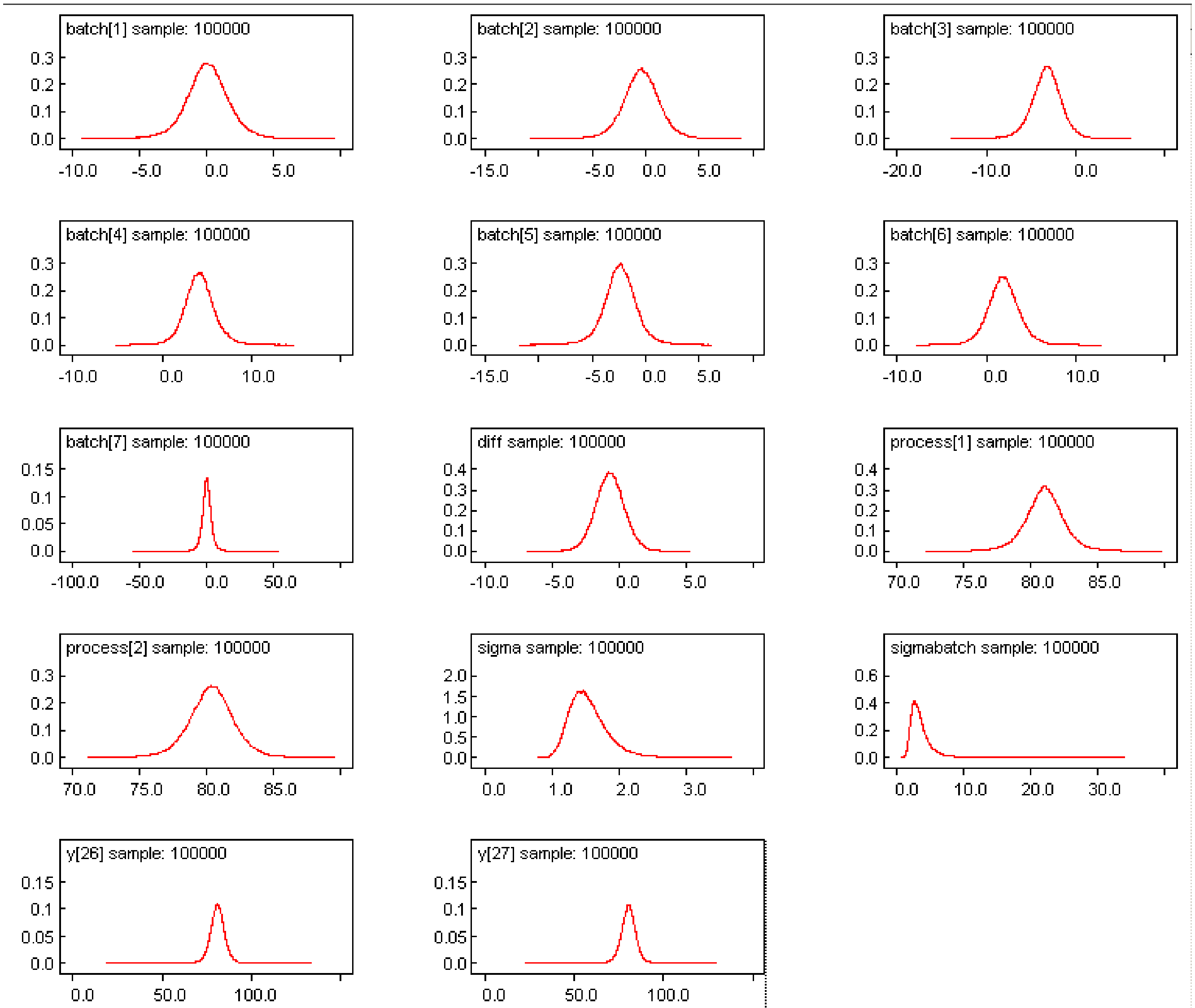
$$y_{xj} = \beta_0 + \beta_1 x + \varepsilon_{xj} \qquad (**)$$

where β_0 and $\beta_1 \ge 0$ are unknown parameters and the ε_{xj} are iid $\mathrm{N}(0, \sigma^2)$. What prior distribution would you use with model (**)? (Fully specify this prior.)

WinBUGS Session Summary for Problem #3

```
model {

for (i in 1:2) {
  process[i] ~ dnorm (0,.0001)
  }

diff<-process[2]-process[1]

logsigb ~ dflat()
taubatch<- exp(-2*logsigb)
sigmabatch<- exp(logsigb)

for (j in 1:7) {
  batch[j] ~ dnorm (0,taubatch)
  }

logsig~dflat()
tau<-exp(-2*logsig)
sigma<- exp(logsig)

for (l in 1:27) {
  mu[l]<-process[p[l]]+batch[b[l]]
  }

for (l in 1:27) {
  y[l]~ dnorm (mu[l],tau)
  }

}

list(b=c(1,1,1,1,2,2,3,3,3,4,4,4,5,5,5,5,5,5,5,5,6,6,6,6,6,7,7),
p=c(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,1,2),
y=c(82.72, 78.31, 82.20, 81.18, 80.06, 81.09, 78.71, 77.48, 76.06, 87.77, 84.42, 84.82,
78.61, 77.47, 77.80, 81.58, 77.50, 78.73, 78.23, 76.40, 81.64, 83.04, 82.40, 81.93, 82.96,NA,NA))
```

node	mean	sd	MC error	2.5%	median	97.5%	start	sample
batch[1]	0.07094	1.646	0.03319	-3.263	0.06109	3.401	1	100000
batch[2]	-0.3905	1.741	0.03172	-3.9	-0.3904	3.09	1	100000
batch[3]	-3.279	1.699	0.03202	-6.806	-3.239	-0.00156	1	100000
batch[4]	4.225	1.7	0.03279	0.949	4.181	7.757	1	100000
batch[5]	-2.388	1.579	0.03368	-5.621	-2.382	0.7832	1	100000
batch[6]	1.92	1.783	0.03465	-1.578	1.892	5.568	1	100000
batch[7]	0.00336	3.712	0.01179	-7.536	0.01809	7.473	1	100000
diff	-0.6633	1.079	0.007092	-2.775	-0.6716	1.49	1	100000
process[1]	81.02	1.531	0.03331	77.92	81.02	84.15	1	100000
process[2]	80.36	1.717	0.03484	76.87	80.37	83.76	1	100000
sigma	1.541	0.2782	0.001303	1.112	1.502	2.194	1	100000
sigmabatch	3.411	1.476	0.01427	1.641	3.076	7.198	1	100000
y[26]	81.02	4.306	0.03525	72.44	81.02	89.57	1	100000
y[27]	80.36	4.373	0.03638	71.57	80.39	88.96	1	100000



`WinBUGS` **Session Summary for Problem #4**

```
model {

  mu1 ~ dnorm(0,.0001)
  mu[1] <- mu1

  for (i in 2:6) {
  delta[i] ~ dexp(.4)
    }

  for (i in 2:6) {
  mu[i] <- mu[i-1]+delta[i]
   }

  logsigma ~ dflat()
  sigma <- exp(logsigma)
  tau <- exp(-2*logsigma)
```

```
for (j in 1:N) {
   y[j] ~ dnorm(mu[group[j]],tau)
   }

}
```

list(N=18,group=c(1,1,1,2,2,2,3,3,3,4,4,4,5,5,5,6,6,6),
y=c(0.9835899, -0.5087186, 1.0450089, 0.6815755, -2.1739497, 1.0464128,
1.3717484, 1.1350734, 0.3384970, 6.5645035, 5.0648255, 6.0209295,
6.5766160, 5.8730637, 7.4934093, 7.8030626, 8.2207331, 6.7444797))

list(mu1=0,logsigma=0)

node	**mean**	**sd**	**MC error**	**2.5%**	**median**	**97.5%**	**start**	**sample**
delta[2]	0.4111	0.373	0.001892	0.0121	0.3086	1.376	1001	200000
delta[3]	0.8161	0.5532	0.002906	0.04305	0.7335	2.097	1001	200000
delta[4]	4.544	0.7616	0.004405	2.912	4.59	5.917	1001	200000
delta[5]	0.9397	0.6133	0.002733	0.05471	0.8552	2.333	1001	200000
delta[6]	0.9577	0.614	0.001835	0.05769	0.8819	2.339	1001	200000
mu[1]	-0.06119	0.4578	0.002214	-1.031	-0.03969	0.7845	1001	200000
mu[2]	0.3499	0.4272	0.001684	-0.4867	0.3466	1.209	1001	200000
mu[3]	1.166	0.5215	0.002508	0.2416	1.13	2.292	1001	200000
mu[4]	5.71	0.5413	0.002383	4.552	5.742	6.692	1001	200000
mu[5]	6.65	0.4842	0.001246	5.69	6.652	7.601	1001	200000
mu[6]	7.608	0.538	0.001037	6.607	7.587	8.729	1001	200000
sigma	1.05	0.2209	8.694E-4	0.721	1.015	1.576	1001	200000