1. Suppose that $X_1, X_2, \ldots, X_n$ are iid exponential with mean $\beta > 0$, i.e. have marginal pdf

$$f(x | \beta) = I[x > 0] \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right)$$

on $\mathbb{R}$. Sadly, all one can observe are “right-censored” versions of the $X_i$, $Y_i = X_i \cdot I[X_i < T] + T \cdot I[X_i \geq T]$ for some fixed (known) constant $T$, and inference about $\beta$ must be based on $Y_1, Y_2, \ldots, Y_n$. ($X_i$ is censored with $\beta$-probability $\exp\left(-\frac{T}{\beta}\right)$, and in such a case $Y_i = T$.) Define

$$n_r = \text{the number of } Y_i \text{ that are } T \quad \text{and} \quad n_a = n - n_r$$

The $Y_i$ are neither discrete nor continuous, so the simple definitions of likelihood used in Stat 543 must be extended to cover this case. When this is done appropriately, a likelihood turns out to be

$$L_n(\beta) = \frac{1}{\beta^{n_a}} \exp\left(-\sum y_i / \beta\right)$$

a) A likelihood for $\gamma = 1/\beta$ is $L_n^*(\gamma) = \gamma^{n_a} \exp\left(-\gamma \sum y_i\right)$. A Bayesian uses an exponential prior distribution with mean 1 for $\gamma$. Set up completely but do not evaluate a ratio of definite integrals giving this person’s (squared error loss) estimator for $\gamma$ based on $Y_1, Y_2, \ldots, Y_n$.

b) Find the maximum likelihood estimator of $\beta$.
c) Argue that your estimator from b) is consistent for $\beta \in (0, \infty)$ (Use Stat 542 facts about convergence in probability and/or distribution. You may use without proof the fact that $E_{\beta} Y_i = \beta (1 - \exp(-T/\beta))$.

In a case where $n = 20$ and $T = 5$, it turns out that $n_u = 10$ and $\sum y_i = 80$.

d) What are approximate 90% confidence limits for $\beta$ in this case, based on your answer to part b) and the “observed Fisher information” in the sample?

The figure below is a plot of the log-likelihood $l_{20}(\beta) = \ln \left( L_{20}(\beta) \right)$ for the values of $T, n_u$, and $\sum y_i$ indicated above. Use it to answer e) and f) on the next page.
e) What are approximate 90% confidence limits for $\beta$ (different from those in part d)) based on the figure? Explain.

f) If I wish to test $H_0: \beta = 17$ versus $H_0: \beta \neq 17$ using $\alpha \approx .05$, does the figure indicate acceptance or does it indicate rejection of the null hypothesis? Explain.

In parts g) and h) below, you may use the fact that (after making an appropriate definition for this case where $Y_1$ is neither discrete nor continuous) the Fisher information in any single censored observation is

$$I_1(\beta) = \frac{1}{\beta^2} \left( 1 - \exp \left( -\frac{T}{\beta} \right) \right)$$

be

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<th>Plan A</th>
<th>Plan B</th>
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<tr>
<td>$n = 80$ and $T = 4$</td>
<td>$n = 40$ and $T = 10$</td>
<td>$n = 20$ and $T = 22$</td>
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Estimation of $\beta$ will be of interest in a future study. Investigators can choose $n$ and $T$. Three $(n,T)$ pairs are under consideration (as requiring about the same experimental effort/total resources). These are:

Investigators will choose between the plans on the basis of a “pre-study best guess” at the value of $\beta$. Which plan seems best if the “pre-study best guess”/“planning value” is $\beta \approx 18$? Justify your answer quantitatively. (Even a correct guess without justification gets no credit.)
h) Consider estimation of the variance of an uncensored observation, $\beta^2$ (still on the basis of $n$ censored observations). What is the Cramér-Rao lower bound on the variance of an unbiased estimator of $\beta^2$?

2. For $\theta \in (0, \infty)$, suppose that $X_1, X_2, \ldots, X_n$ are iid $U(0, \theta)$, that is, with marginal pdf on $\mathbb{R}$

$$f(x | \theta) = \theta^{-1} I[0 \leq x \leq \theta]$$

consider the MLE of $\theta$,

$$\hat{\theta}_n = \max \{X_1, X_2, \ldots, X_n\}$$

a) Prove that in this “non-regular” problem $n(\theta - \hat{\theta}_n) \xrightarrow{L} \text{Exp}(\theta)$ (exponential with mean $\theta$). (Hint: Evaluate $P_\theta \left[ n(\theta - \hat{\theta}_n) > t \right]$ and remember that $\lim_{s \to \infty} \left(1 + \frac{a}{s}\right)^s = \exp(a)$.)

b) Argue carefully that $n \frac{\hat{\theta}_n}{(n - \ln(20))}$ can be used as an approximately 95% upper confidence bound for $\theta$. 
3. Suppose that a discrete random variable $X$ has pmf $f(x \mid \theta)$ specified below for $\theta \in \{1, 2, 3, 4\}$.

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a) Carefully derive the size $\alpha = .4$ likelihood ratio test of $H_0: \theta = 1$ or 2 versus $H_a: \theta = 3$ or 4. (Show your reasoning!)

b) A Bayesian (using 0-1 loss) wishes to test $H_0: \theta = 1$ vs $H_a: \theta \neq 1$ and adopts a prior distribution with pmf specified by $g(1) = 1/2, g(2) = 1/6, g(3) = 1/6$, and $g(4) = 1/6$. What test is optimal for this person? (Specify $\phi(x)$ for all 5 values of $x$.)
4. For \( \eta \in \mathbb{R} \), consider \((X, Y)\) with (joint) pdf on \( \mathbb{R}^2 \)
\[
f(x, y | \eta) = I[0 < x < 1, 0 < y < 1]C(\eta)\exp(\eta |x - y|)
\]
(for some normalizing constant \( C(\eta) \)).

a) Suppose that Vardeman wants to evaluate the \( \eta = 2 \) mean of \( X \) using a Gibbs sampling algorithm. Carefully describe how to do this. (What conditionals does he sample from? You only need to specify them up to a multiplicative constant. Then what does he do with the generated sequence of \((X, Y)\)'s?)

b) Identify a UMP size \( \alpha = .25 \) test of \( H_0: \eta \leq 0 \) vs \( H_a: \eta > 0 \) based on a single observation \((X, Y)\) and argue carefully that your test is indeed UMP of its size.

c) If \((X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)\) are iid with marginal pdf \( f(x, y | \eta) \), find a minimal sufficient statistic and argue carefully that it is indeed minimal sufficient.