Theorem 1 Suppose $f(x|\theta) \ \forall \theta \in \Theta$ is either a probability density for $X$ on $\mathbb{R}^k$ or a probability mass function for $X$, and $T(X)$ is sufficient for $\theta$. If the existence of a positive number $k(x,y)$ such that $f(y|\theta) = k(x,y) f(x|\theta) \ \forall \theta \in \Theta$ implies that $T(y) = T(x)$, then $T(X)$ is minimal sufficient.

**Proof.** Let $S$ be any other sufficient statistic. \exists nonnegative functions $g(s, \theta)$ and $h(x)$ such that
\[
f(x|\theta) = g(S(x), \theta) h(x)
\]
and we may wolog assume that $h(x) > 0$. (If not, I start over after redefining the observation space by throwing out possible values where $h(x) = 0$. The set of points I thus throw away have probability 0 \forall \theta \in \Theta.)

Suppose that $S(y) = S(x)$. For all $\theta$
\[
f(y|\theta) = g(S(y), \theta) h(y) = g(S(x), \theta) h(y) = g(S(x), \theta) h(x) \left( \frac{h(y)}{h(x)} \right)
\]
So for $k(x, y) = h(y) / h(x)$ we see that $f(y|\theta) = k(x, y) f(x|\theta) \ \forall \theta \in \Theta$ and thus that $T(y) = T(x)$. That is,
\[
S(y) = S(x) \implies T(y) = T(x) \quad (*)
\]
We need to show that this implies the existence of a function $q(s)$ such that $T(x) = q(S(x))$.

For each $s \in \text{Range}(S)$, let $x_s$ be a possible value of $X$ such that
\[
S(x_s) = s
\]
Define $q(s) = T(x_s)$. This definition is unambiguous because of $(*)$. Then
\[
q(S(x)) = T(x_{S(x)}) = T(x)
\]
where the last equality follow from the fact that $S(x) = S(x_{S(x)})$ and implication $(*)$. □