Theorem 2 (Bahadur’s Theorem) Suppose that $T(X)$ taking values in $\mathbb{R}^k$ is sufficient for $\theta$ and boundedly complete. Then $T(X)$ is minimal sufficient.

Proof. Without loss of generality, we may assume that each coordinate of $T(X) = (T_1(X), T_2(X), ..., T_k(X))$ takes values in $(0, 1)$. If not, for example

$$T^*(X) = \left( \frac{1}{1 + \exp(T_1(X))}, ..., \frac{1}{1 + \exp(T_k(X))} \right)$$

is equivalent to $T(X)$, and takes values in $(0, 1)^k$.

Let $S(X)$ be any other sufficient statistic. We want to show that $T(X)$ can be realized as a function of $S(X)$. Define

$$H_i(s) = \mathbb{E}[T_i(X) | S(X) = s]$$

(note that by sufficiency, the expectation here doesn’t depend on $\theta$). Further, let

$$L_i(t) = \mathbb{E}[H_i(S(X)) | T(X) = t] ,$$

(and again note that by sufficiency, the expectation doesn’t depend on $\theta$). We will show that $P_\theta[T_i(X) = H_i(S(X))] = 1 \forall \theta$, and then have the desired conclusion.

Now the fact that each $T_i(X)$ takes values in $(0, 1)$ implies that $0 \leq H_i(s) \leq 1$ and that $0 \leq L_i(t) \leq 1$. Note too that

$$\mathbb{E}_\theta T_i(X) = \mathbb{E}_\theta (\mathbb{E}[T_i(X) | S(X)]) = \mathbb{E}_\theta H_i(S(X)) = \mathbb{E}_\theta (\mathbb{E}[H_i(S(X)) | T(X)]) = \mathbb{E}_\theta L_i(T(X))$$

So

$$\mathbb{E}_\theta (T_i(X) - L_i(T(X))) = 0 \ \forall \theta .$$

Bounded completeness then implies that

$$P_\theta[T_i(X) = L_i(T(X))] = 1 \forall \theta . \quad (1)$$

So

$$\mathbb{E}[L_i(T(X)) | S(X) = s] = \mathbb{E}[T_i(X) | S(X) = s] = H_i(s) . \quad (2)$$
Then for any $\theta$

\[ \text{Var}_\theta L_i(T(X)) = E_\theta \text{Var} [L_i(T(X)) | S(X)] + \text{Var}_\theta E[L_i(T(X)) | S(X)] \]

\[ = E_\theta \text{Var} [L_i(T(X)) | S(X)] + \text{Var}_\theta H_i(S(X)) \]

\[ = E_\theta \text{Var} [H_i(S(X)) | T(X)] + \text{Var}_\theta E[H_i(S(X)) | T(X)] \]

\[ = E_\theta \text{Var} [H_i(S(X)) | T(X)] + \text{Var}_\theta L_i(T(X)) \].

(The second equality above follows from (2).) Now the fact that the random variables \(\text{Var} [L_i(T(X)) | S(X)]\)
and \(\text{Var} [H_i(S(X)) | T(X)]\) are both nonnegative implies that both of the expectations on the right side of
the last line above are nonnegative. This in turn implies that they are both 0, which in turn implies that

\[ P_\theta [\text{Var} [L_i(T(X)) | S(X)] = 0] = 1 \]

so that

\[ P_\theta (L_i(T(X) = E[L_i(T(X)) | S(X)]) = 1 \].

But again \(E[L_i(T(X)) | S(X)] = H_i(S(X))\), so by (1)

\[ P_\theta[T_i(X) = L_i(T(X)) = H_i(S(X))] = 1 \].