

Hint for Problem 1.2.14 of B&D

The general story in this problem is this:

For $X_1, X_2, X_3, \dots, X_n, X_{n+1}$ iid $f(x|\theta)$ and $\theta \sim g(\theta)$, the joint density of the observables and parameter is

$$f(\mathbf{x}, \theta) = g(\theta) \prod_{i=1}^{n+1} f(x_i|\theta)$$

and from this, one can (in theory) get the marginal of the vector of observables \mathbf{X} by integrating out θ , i.e.

$$f_g(\mathbf{x}) = \int g(\theta) \prod_{i=1}^{n+1} f(x_i|\theta) d\theta$$

THEN, from this one can contemplate finding the conditional distribution of $X_{n+1}|X_1, X_2, \dots, X_n$ that B&D call the predictive distribution (note that you don't get this "for free" in terms of assumptions ... the form of f_g depends on g).

Now actually carrying out the above program is going to be very unpleasant except for "convenient" cases. In fact, the only multivariate distributions that are really "convenient" are the MVN distributions. Thankfully, (and not surprisingly) this problem uses them. That is, think of writing

$$X_i = \theta + \epsilon_i$$

for the ϵ_i iid $N(0, \sigma_0^2)$ independent of $\theta \sim N(\theta_0, \tau_0^2)$. Then

$$\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \\ X_{n+1} \\ \theta \end{pmatrix} \sim \text{MNV}(\theta_0 \mathbf{1}, \Sigma)$$

for an appropriate Σ that you can figure out easily enough, since finding variances and covariances is straightforward. ($\text{Var}X_i = \sigma_0^2 + \tau_0^2, \text{Cov}(X_i, X_j) = \tau_0^2, \text{Cov}(X_i, \theta) = \tau_0^2$). From this, the marginal of $X_1, X_2, X_3, \dots, X_n, X_{n+1}$ is obvious (also MVN) and you can use facts about conditionals of MVN distributions to get the conditional distribution of $X_{n+1}|X_1, X_2, \dots, X_n$.

In doing the above, you will probably need to use a matrix fact about the inverse of a certain patterned matrix. The covariance matrix of \mathbf{X} (and of any subset of the entries of \mathbf{X}) is of the form

$$a\mathbf{I} + b\mathbf{J}$$

for \mathbf{J} a matrix of 1's. The inverse of such a matrix has the form

$$\alpha\mathbf{I} + \beta\mathbf{J}$$

for $\alpha = \frac{1}{a}$ and $\beta = -\frac{b}{a(a+kb)}$, where k is the dimension of the (square) matrix \mathbf{I} (or \mathbf{J}).