Stat 543 Assignment 11
Asymptotics of the LRT and Posterior Distributions, and Wald and Score Tests

1. (More Estimation in a Zero-Inflated Poisson Model) Consider the situation of Problem 1 of Assignment 10, and in particular, inference based on the $n = 20$ observations Vardeman simulated from the distribution.

   (a) Find a large sample 90% joint confidence region for (p, λ) based on the loglikelihood function (based on inverting LRT's). Plot this in the (p, λ)-plane and to the extent possible, compare it to the elliptical region you found in Assignment 10.

   (b) Note that for a fixed value of λ,

   $$p = \frac{\frac{n_0}{n} - 1}{\exp(-\lambda) - 1}$$

   maximizes the likelihood. Use this fact to find and plot the profile loglikelihood for λ. Use this plot and make an approximate 90% confidence interval for λ. How does this interval compare to the one you found in part e) of Problem 1 from Assignment 10?

2. Consider again the model of Problem 2 of Assignment 10. Below are $n = 20$ observations that Vardeman simulated from this model.

   $$1.36, 1.35, 0.78, 1.85, 2.32, 0.55, 1.07, -0.57, -0.38, 0.25,$$
   $$-0.36, 1.71, 1.40, 0.46, 3.16, -0.78, 0.69, -0.03, 1.26, 0.44$$

   (a) Plot the loglikelihood for this sample. What, approximately, is the maximum likelihood estimate for α?

   (b) If you wished to test the hypothesis H_0:$\alpha = .4$ with Type I error probability .1, what would be your decision here? Carefully explain. (Use a likelihood ratio test).

   (c) Use the plot from a) and make an approximate 90% confidence interval for α based on the likelihood function (based on inverting LRT's). Use the method of f) of Problem 2 on Assignment 10 and make another approximate 90% interval. How do these 2 intervals compare?

3. Problems 1.2.4 and 1.2.5 of B&D.

4. Consider Bayesian inference for the binomial parameter p. In particular, for sake of convenience, consider the Uniform $(0, 1)$ (Beta(α, β) for $\alpha = \beta = 1$) prior distribution.

   (a) It is possible to argue from reasonably elementary principles that in this binomial context, where $\Theta = (0, 1)$, the Beta posteriors have a consistency property. That is, simple arguments can be used to show that for any fixed $p_0$and any $\epsilon > 0$, for $X_n \sim$binomial (n, p_0), the random variable

$$Y_n = \int_{p_0-\epsilon}^{p_0+\epsilon} \frac{1}{B(\alpha + X_n, \beta + (n - X_n))} p^{\alpha+X_n-1}(1-p)^{\beta+(n-X_n)-1}dp$$

(which is the posterior probability assigned to the interval $(p_0-\epsilon, p_0+\epsilon)$) converges in p_0 probability to 1 as $n \to \infty$. This part of the problem is meant to lead you through this argument. Let $\epsilon > 0$ and $\delta > 0$.

i) Argue that there exists m such that if $n \geq m$, $\left|\frac{x_n}{n} - \frac{\alpha+x_n}{\alpha+\beta+n}\right| < \frac{\epsilon}{3}\ \forall x_n = 0, 1, ..., n$.

ii) Note that the posterior variance is $\frac{(\alpha+x_n)(\beta+n-x_n)}{(\alpha+\beta+n)^2(\alpha+\beta+n+1)}$. Argue there is an m' such that if $n \geq m'$ the probability that the posterior assigns to $\left(\frac{\alpha+x_n}{\alpha+\beta+n} - \frac{\epsilon}{3}, \frac{\alpha+x_n}{\alpha+\beta+n} + \frac{\epsilon}{3}\right)$ is at least $1-\delta\ \forall x_n = 0, 1, ..., n$.

iii) Argue that there is an m'' such that if $n \geq m''$ the p_0 probability that $\left|\frac{X_n}{n} - p_0\right| < \frac{\epsilon}{3}$ is at least $1-\delta$.

Then note that if $n \geq \max(m, m', m'')$ i) and ii) together imply that the posterior probability assigned to $\left(\frac{x_n}{n} - \frac{2\epsilon}{3}, \frac{x_n}{n} + \frac{2\epsilon}{3}\right)$ is at least $1-\delta$ for any realization x_n. Then provided $\left|\frac{x_n}{n} - p_0\right| < \frac{\epsilon}{3}$ the posterior probability assigned to $(p_0 - \epsilon, p_0 + \epsilon)$ is also at least $1-\delta$. But iii) says this happens with p_0 probability at least $1-\delta$. That is, for large n, with p_0 probability at least $1-\delta$, $Y_n \geq 1-\delta$. Since δ is arbitrary, (and $Y_n \leq 1$) we have the convergence of Y_n to 1 in p_0 probability.

(b) Vardeman intends to argue in class that posterior densities for large n tend to look normal (with means and variances related to the likelihood material). The posteriors in this binomial problem are Beta $(\alpha + x_n, \beta + (n - x_n))$ (and we can think of $X_n \sim$Bi (n, p_0) as derived as the sum of n iid Bernoulli (p_0) variables). So we ought to expect Beta distributions for large parameter values to look roughly normal. To illustrate this do the following. For $\rho = .3$ (for example ... any other value would do as well), consider the Beta $(\alpha + n\rho, \beta + n(1-\rho))$ (posterior) distributions for $n = 10, 20, 40$ and 100. For $p_n \sim$Beta $(\alpha + n\rho, \beta + n(1-\rho))$ plot the probability densities for the variables

$$\sqrt{\frac{n}{\rho(1-\rho)}}\,(p_n - \rho)$$

on a single set of axes along with the standard normal density. Note that if W has pdf $f(\cdot)$, then $aW + b$ has pdf $g(\cdot) = \frac{1}{a} f\left(\frac{\cdot - b}{a}\right)$. (Your plots are translated and rescaled posterior densities of p based on possible observed values $x_n = .3n$.) If this is any help in doing this plotting, Vardeman tried to calculate values of the Beta function using MathCad and got the following: $(B(4,8))^{-1} = 1.32 \times 10^3$, $(B(7,15))^{-1} = 8.14 \times 10^5$, $(B(13,29))^{-1} = 2.291 \times 10^{11}$ and $(B(31,71))^{-1} = 2.967 \times 10^{27}$.

5. Suppose that X, Y and Z are independent binomial variables, $X \sim$bin(n, p_1), $Y \sim$bin(n, p_2) and $Z \sim$bin(n, p_3). For the parameter space (for (p_1, p_2, p_3)) $\Theta = [0, 1]^3$, we will consider testing H$_0$:$p_1 = p_2 = p_3$ based on (X, Y, Z).

(a) Find the general forms of the likelihood ratio tests, the Wald tests and the score tests of this hypothesis.

(b) Use the fact that the parameter space here is basically 3-dimensional while Θ_0 is basically 1-dimensional so that there are 2 independent constraints involved and the limiting χ^2 distributions of the test statistics thus have $\nu = 2$ associated degrees of freedom to actually carry out these tests with $\alpha \approx .05$ if $X = 33, Y = 53$ and $Z = 59$, all based on $n = 100$.