1. Consider the simple discrete distribution specified by the probability mass function given below in tabular form for \( \theta \in \Theta = [0, .302] \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x</td>
<td>\theta) )</td>
<td>( \theta )</td>
<td>( 2\theta )</td>
<td>( \theta^2 )</td>
</tr>
</tbody>
</table>

Consider first estimation of \( \theta \) based on a single observation from this distribution, \( X \).

a) Show explicitly that the mean of the score function here is 0 (for all \( \theta \)).

b) Write out (but do not take the time to simplify) an explicit expression for the Fisher information in \( X \) about \( \theta \). (As it turns out, I believe that \( I(\theta) = \frac{3+4\theta-6\theta^2-3\theta^3}{\theta(1-3\theta-\theta^2)} \).)

c) One possible estimator of \( \theta \) is

\[
\hat{\theta}(X) = \begin{cases} 
.302 & \text{if } X = 0 \\
.232 & \text{if } X = 1 \\
.016 & \text{if } X = 2 \\
0 & \text{if } X = 3 
\end{cases}
\]

Find a better estimator of \( \theta \) (in terms of MSE) and argue carefully that it is better.

d) Consider now a Bayesian approach to estimation of \( \theta \) with a prior distribution uniform on the interval \([0, .2]\). Suppose that \( X = 2 \).

i) What is the posterior probability that \( \theta > .25 \) (based on the outcome \( X = 2 \))? 

ii) Find the posterior mean of \( \theta \) (based on the outcome \( X = 2 \)).

iii) Find a 90% highest posterior density credible region for \( \theta \) (based on the outcome \( X = 2 \)).

iv) Carefully describe how you could use the rejection algorithm and a stream of iid Uniform \((0, 1)\) random variables to create a sequence \( \{\theta^*_n\} \) of iid observations from the posterior distribution (based on the outcome \( X = 2 \)).

Now consider the possibility of estimating \( \theta \) based on \( n \) iid observations \( X_1, X_2, ..., X_n \). Henceforth suppose \( \delta_n(X) \) is the MLE of \( \theta \). (Don't try to actually find the form of the MLE of \( \theta \).)

e) Using again the prior distribution for \( \theta \) described in d), what, in qualitative terms, would you expect to happen (as \( n \to \infty \)) to the posterior distributions of \( \theta \)

i) if in fact \( \theta = .15 \)?

ii) if in fact \( \theta = .25 \)?
The graph attached to this exam is for the loglikelihood in this problem, based on a sample of \( n = 100 \) observations with the frequency distribution below:

<table>
<thead>
<tr>
<th>( x )</th>
<th>frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>19</td>
</tr>
<tr>
<td>1</td>
<td>34</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
</tr>
</tbody>
</table>

f) Use the graph and give a large sample 90% confidence interval for \( \theta \) based on the large sample distribution of \( \hat{\theta} \) and the observed (not expected) Fisher information in this sample.

g) Use the graph of the loglikelihood \( \ell_n(\theta) \) and give a large sample 90% confidence interval for \( \theta \) based on the large sample distribution of \( \ell_n(\delta_n(X)) - \ell_n(\theta) \).

2. Consider the problem of estimation in the Weibull family of distributions based on \( n \) iid observations \( X_1, X_2, \ldots, X_n \). For scale and location parameters \( \alpha \) and \( \beta \), the loglikelihood here is:

\[
\ell_n(\alpha, \beta) = n \ln \beta - n \beta \ln \alpha + (\beta - 1) \left( \sum_{i=1}^{n} \ln x_i \right) - \frac{1}{\alpha \beta} \left( \sum_{i=1}^{n} x_i^\beta \right)
\]

and the likelihood equations reduce to

\[
\beta = \left( \frac{\sum_{i=1}^{n} x_i^\beta \ln x_i}{\sum_{i=1}^{n} x_i^\beta} - \frac{n}{\sum_{i=1}^{n} x_i^\beta} \right)^{-1}
\]

and

\[
\alpha = \left( \frac{\sum_{i=1}^{n} x_i^\beta}{\sum_{i=1}^{n} x_i^\beta} \right)^{\frac{1}{\beta}}.
\]

There are no closed form solutions for these equations (i.e. there are no explicit formulas for MLEs here). However, suppose that I want to make a 90% approximate confidence interval for \( \beta \) (the shape parameter). Describe in as much detail as possible how you would go about producing such an interval.