Stat 543 II

The $\theta$-left-truncated Normal ($\mu, 1$) distribution has probability density

$$f(x|\theta, \mu) = I[x \geq \theta] \frac{\phi(x - \mu)}{1 - \Phi(\theta - \mu)}$$

for $I[x \geq \theta]$ the indicator that $x \geq \theta$, $\Phi$ the standard normal cdf and $\phi$ the standard normal density. This distribution has mean

$$\Delta(\theta, \mu) = \mu + \frac{\phi(\theta - \mu)}{1 - \Phi(\theta - \mu)} ,$$

and attached to this exam is a plot of $\Delta(\theta, \mu)$ as a function of $\mu$ for several different values of $\theta$.

Suppose that $X_1, X_2, ..., X_n$ are iid from this density and inference for $(\theta, \mu) \in \mathbb{R}^2$ is under discussion. Let $M_n = \min(X_1, X_2, ..., X_n)$ and $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and denote realized values of $M_n$ and $\bar{X}_n$ by $m_n$ and $\bar{x}_n$ respectively.

a) Argue carefully that the random vector $(M_n, \bar{X}_n)$ is sufficient for $(\theta, \mu)$.

Let $L_n(\theta, \mu)$ be the likelihood function and $l_n(\theta, \mu)$ be the loglikelihood here.

b) Argue carefully that for any fixed $\mu$, $\hat{\theta}_n = m_n$ maximizes $L_n(\theta, \mu)$ as a function of $\theta$.

c) Show that if $\hat{\mu}_n$ maximizes $L_n(m_n, \mu)$, it must be a solution of the equation

$$\bar{x}_n = \Delta(m_n, \mu) .$$

d) Suppose with $n = 25$, one observes $m_n = .5$ and $\bar{x}_n = 1.5$. Find the MLE of $(\theta, \mu)$ based on the attached plot of $\Delta(\theta, \mu)$.

e) Consider testing $H_0 : (\theta, \mu) = (.25,.8)$ using a likelihood ratio test. Write out an explicit formula for the likelihood ratio statistic for this particular model and null hypothesis.

f) As it turns out, the large $n$ null distribution of $2(l_n(M_n, \hat{\mu}_n) - l_n(.25, .8))$ is not $\chi^2$ as would be expected under standard regularity conditions (which don't hold here). Rather, it is $\chi^2_3$. Attached to this exam is a table of $\chi^2$ quantiles. What can you say about the approximate $p$-value for the test considered in e) based on the sample referred to in d)?

g) As it turns out, the $(\theta, \mu)$ large $n$ distribution of $2(l_n(M_n, \hat{\mu}_n) - l_n(M_n, \mu))$ is $\chi^2_1$ (just as if standard regularity conditions held in this model). Attached to this exam is a plot of $l_n(.5, \mu)$ for the sample referred to in d). Use the plot and this fact to find an approximate 90% confidence interval for $\mu$. (The maximum value of $l_n(.5, \mu)$ shown on the plot is $-23.123$.)