Suppose that $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Normal}(\mu, 1)$ are independent and that one observes the random vector $(X, Z)$ for $Z = XY$. (One may think of this situation as one of "noninformative censoring" where unless $X = 1$ one fails to observe $Y$.) The vector parameter for this model is $(p, \mu)$. The hypothesis $H_0 : p\mu \geq 5$ is of interest.

(a) **Find** the mean and variance of $Z$.

(b) A possible test here is

$$\phi(x, z) = \begin{cases} 
1 & \text{if } x = 1 \text{ and } z < 5 - 1.645 \\
0 & \text{otherwise.}
\end{cases}$$

**Find** the power function of this test, $\beta(p, \mu)$, and the size of this test for testing $H_0 : p\mu \geq 5$.

(c) **Argue** very carefully that the test in (b) is a likelihood ratio test of $H_0$ versus $H_a : \text{not } H_0$. (Hint: You should consider $(x, z)$ pairs of three types: $(0, 0), (1, z)$ for $z \geq 5$ and $(1, z)$ for $z < 5$.)

(d) In contrast to the model indicated above, (in this part of the question only) consider the possibility that $Y \sim \text{Normal}(\mu, 1), X = I[Y > 0]$ and $Z = XY$. (This is a case where the censoring is informative. The censoring mechanism carries some information about $\mu$.) **Find** the power function of the test from (b) under this model. (This will be a function of $\mu$ only. $X$ here is still Bernoulli, but it is clearly dependent upon $Y$ and "$p$" here is a function of $\mu$.)

Now for large $n$, suppose that $(X_i, Z_i)$ for $i = 1, ..., n$ are independent, each with distribution described at the top of the page.

(e) Keeping in mind the result of part (a), **give** consistent estimators of both $p$ and $p\mu$.

(f) Based on the CLT, **propose** an approximately size $\alpha = .05$ test of $H_0 : p\mu \geq 5$ versus $H_a : p\mu < 5$ that is a function of $(\bar{X}, \bar{Z})$. (Hint: Consider the large sample distribution of $\bar{Z}$. Be sure your test statistic doesn't depend upon any unknown parameters.)

(g) Carefully **describe** how you could use simulation to approximate the power of your test from (f) when $n = 20$, $p = .5$ and $\mu = 13$. 