Suppose first that $X$ is a discrete random variable with probability mass function $f(x|\theta)$ satisfying standard regularity conditions for $\theta$ in $\Theta$ an open interval in $R$.

(a) Define $I_X(\theta_0)$, the Fisher information in $X$ about the parameter $\theta$, evaluated at $\theta_0 \in \Theta$.

(b) Suppose that for another open interval $\Gamma$ in $R$, the differentiable increasing function $h$ maps $\Gamma$ onto $\Theta$. Consider the (reparameterized) model for $X$ defined by

\[
g(x|\gamma) = f(x|h(\gamma)) \quad \text{for} \quad \gamma \in \Gamma.
\]

Express the Fisher information about the parameter $\gamma$, evaluated at $\gamma_0 \in \Gamma$, in terms of the functions $I_X(\theta)$ and $h(\gamma)$. (Write $h'$ for the derivative of $h$.) Argue carefully that your expression is correct.

Suppose now that $Y_1, Y_2, \ldots, Y_n$ are independent Bernoulli random variables. For known constants $t_1, t_2, \ldots, t_n$ and unknown parameters $\alpha \in R$ and $\beta \in R$ we will suppose that

\[
P[Y_i = 1] = \frac{1}{1 + \exp(- (\alpha + \beta t_i))}.
\]

This is equivalent to assuming that

\[
\ln \left( \frac{P[Y_i = 1]}{1 - P[Y_i = 1]} \right) = \alpha + \beta t_i.
\]

(c) Write out the log-likelihood function, $L(\alpha, \beta)$, for this problem and a pair of equations that will have to be solved in order to find maximum likelihood estimates of the parameters $\alpha$ and $\beta$.

For a particular famous data set with $n = 23$, two figures summarizing a likelihood analysis of this problem are attached to this question. Use them in what follows. It will also be helpful to know that MLE's for this data set are $\hat{\alpha} = 15.044$ and $\hat{\beta} = -.2322$ and $L(15.044, -.2322) = -10.158$.

For $\beta \in R$, let $\alpha^*(\beta)$ be a maximizer of $L(\alpha, \beta)$ considered as a function of $\alpha$. Define

\[
L^*(\beta) = L(\alpha^*(\beta), \beta).
\]

Figure 1 is a plot of this function.

(d) Note that $\beta = 0$ in either (1) or (2) is the case where $P[Y = 1]$ doesn't depend upon $t$. Is $H_0 : \beta = 0$ a plausible hypothesis in the light of the Figure 1? Explain. (Hint: Consider a likelihood ratio test of $H_0$.)

(e) Give an approximate 90% confidence interval for $\beta$ based on Figure 1.

In the real problem, the quantity $p_{31} \overset{\text{def}}{=} \frac{1}{1 + \exp(- (\alpha + 31 \beta))}$ was of vital interest. (This is the "success probability" for a $Y$ with $t = 31$.) It is possible to parameterize this problem in terms of $p_{31}$ and, say, $p_{72} \overset{\text{def}}{=} \frac{1}{1 + \exp(- (\alpha + 72 \beta))}$ (the "success probability" for a $Y$ with $t = 72$).

(f) Give MLE's of $p_{31}$ and $p_{72}$ and argue carefully that your values are correct.

(g) The hope was that $p_{31}$ was in fact small. Figure 2 is a plot outlining the region in $(p_{31}, p_{72})$-space with log-likelihood exceeding $-12.4605$. Carefully interpret Figure 2 in the light of the hope about $p_{31}$.