Throughout this question, for $\theta_1 > 0$ and $\theta_2 \in \mathcal{R}$, let $f(x|\theta_1, \theta_2)$ be the two parameter exponential probability density

$$f(x|\theta_1, \theta_2) = \begin{cases} \theta_1 \exp(-\theta_1(x - \theta_2)) & \text{for } x \geq \theta_2 \\ 0 & \text{otherwise} \end{cases}.$$  

(a) Show that if $U \sim \text{Uniform}(0, 1)$, then $Y = -\ln U$ has the "standard" exponential distribution (i.e. the exponential distribution with $\theta_1 = 1$ and $\theta_2 = 0$).

(b) Based on the fact in (a) and the availability of a uniform random number generator, how do you suggest simulating a random sample of size $n$ from the Exponential($\theta_1, \theta_2$) distribution? (Here we suppose that $\theta_1$ and $\theta_2$ are known constants.)

Let $X_1, X_2, ..., X_n$ be iid with marginal density $f(x|\theta_1, \theta_2)$.

(c) Find the (joint) maximum likelihood estimators of $\theta_1$ and $\theta_2$ based on $X_1, X_2, ..., X_n$.

(d) Argue carefully that your MLE of $\theta_1$ from (c) is consistent.

(e) Give the form of the likelihood ratio tests of

  i) $H_0: \theta_1 = \theta_1^0$ vs $H_a: \text{not } H_0$

  and

  ii) $H_0: \theta_2 = \theta_2^0$ vs $H_a: \text{not } H_0$

(Write complete formulas for the test statistics and indicate what kinds of values of these will cause rejection of the null hypotheses. But you need NOT simplify the forms of your statistics nor speculate how to choose cut-offs to get $\alpha$ level tests.)

For each $i = 1, 2, ..., n$ let

$$X_i^* = X_i \text{ rounded to the nearest integer}$$

and now suppose that one gets to observe not the $X_i$'s but rather the $X_i^*$'s. In fact, suppose that the frequency distribution of what is observed in a sample of $n = 20$ is

<table>
<thead>
<tr>
<th>$x^*$</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>7</td>
<td>8</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
(f) Write out an appropriate likelihood function $L^*(\theta_1, \theta_2)$ based on these data. (Hint: For $x^* = 5, 6, 7, 8$ and $9$, what are $P_{\theta_1, \theta_2}[X^* = x^*$] ? Be careful. You will need one prescription for the likelihood in cases where $\theta_2 < 4.5$ and another for cases where $\theta_2 \geq 4.5$.)

As a matter of fact, the likelihood function referred to in (f) is maximized at $\theta_1 = 0.894$ and $\theta_2 = 5.018$. With $\mathcal{L}(\theta_1, \theta_2) = \ln L^*(\theta_1, \theta_2)$, $\mathcal{L}(0.894, 5.018) = -27.833$ and second partial derivatives of $\mathcal{L}$ evaluated at $(0.894, 5.018)$ are

$$\frac{\partial^2 \mathcal{L}}{\partial \theta_1^2} = -23.9, \quad \frac{\partial^2 \mathcal{L}}{\partial \theta_2^2} = -29.7 \quad \text{and} \quad \frac{\partial^2 \mathcal{L}}{\partial \theta_1 \partial \theta_2} = 16.0.$$  

(g) Give maximum likelihood estimates of $\theta_1$ and $\theta_2$ and appropriate standard errors for the estimates. Do the data suggest that maximum likelihood estimators based on $X_i^*$'s will be positively correlated or negatively correlated? Explain.

(h) Attached to this question is a contour plot of $\mathcal{L}(\theta_1, \theta_2)$. Use it and test the hypothesis $H_0: \theta_1 = 1.30$ and $\theta_2 = 4.90$ vs $H_a: \theta_1 \neq 1.30$ with $\alpha \approx 0.05$. (There is a table of $\chi^2$ percentage points also attached to this question.)