4.7

We will measure time in minutes past 8 A.M. So $X \sim \text{uniform}(0,30), Y \sim \text{uniform}(40,50)$ and the joint pdf is $1/300$ on the rectangle $(0, 30) \times (40, 50)$.

\[
P(\text{arrive before 9 A.M.}) = P(X + y < 60) = \int_{40}^{50} \int_{0}^{60-y} \frac{1}{300} \, dx \, dy = \frac{1}{2}
\]

4.10

a. The marginal distribution of $X$ is $P(X = 1) = P(X = 3) = 1/4$ and $P(X = 2) = 1/2$.

The marginal distribution of $Y$ is $P(Y = 2) = P(Y = 3) = P(Y = 4) = 1/3$. But

\[P(X = 2, Y = 3) = 0 \neq \left( \frac{1}{2} \right) \left( \frac{1}{3} \right) = P(X = 2)P(Y = 3).
\]

Therefore, the random variables are not independent.

b. The distribution that satisfies $P(U = x, V = y) = P(U = x)P(V = y)$ where $U \sim X$ and $V \sim Y$ is

<table>
<thead>
<tr>
<th>Distribution of U and V</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>p(V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>1/12</td>
<td>1/6</td>
<td>1/12</td>
<td>1/3</td>
</tr>
<tr>
<td>V</td>
<td>1/12</td>
<td>1/6</td>
<td>1/12</td>
<td>1/3</td>
</tr>
<tr>
<td>4</td>
<td>1/12</td>
<td>1/6</td>
<td>1/12</td>
<td>1/3</td>
</tr>
<tr>
<td>p(U)</td>
<td>1/4</td>
<td>1/2</td>
<td>1/4</td>
<td></td>
</tr>
</tbody>
</table>

4.11

The support of the distribution of $(U,V)$ is $\{(u,v) : u = 1, 2, 3, \ldots; v = u + 1, u + 2, u + 3, \ldots\}$. This is not a cross-product set. Therefore, $U$ and $V$ are not independent. More simply, if we know $U = u$, then we know $V > u$.

4.22

\[
F_{U,V}(u,v) = F_{U,V}(U \leq u, V \leq v)
= F_{U,V}(aX + b \leq u, cY + d \leq v)
= F_{X,Y}(X \leq \frac{u-b}{a}, Y \leq \frac{v-d}{c})
\]
\[ f_{U,V}(u,v) = \frac{\partial^2 F_{XY}(\frac{u-b}{a}, \frac{v-d}{c})}{\partial u \partial v} = f_{X,Y}(\frac{u-b}{a}, \frac{v-d}{c}) \cdot |J| \]
\[ = \frac{1}{ac} f_{X,Y}(\frac{u-b}{a}, \frac{v-d}{c}) \]
where
\[ |J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/a & 0 \\ 0 & 1/c \end{vmatrix} = \frac{1}{ac} \]

### 4.30

**a.**

\[ EY = E[E(Y|X)] = E[X] = \int_0^1 x \, dx = \frac{1}{2} \]
\[ VarY = Var[E(Y|X)] + E[Var[Y|X]] = Var(X) + E(X^2) = \frac{1}{12} + \frac{1}{3} = \frac{5}{12} \]
\[ EXY = E[E(XY|X)] = E(XE(Y|X)) = EX^2 = \frac{1}{3} \]
\[ Cov(X,Y) = EXY - EXEY = \frac{1}{3} - \left( \frac{1}{2} \right)^2 = \frac{1}{12} \]

**b.** let \( U = Y/X \) and \( V = X \). Then \( X = V \), \( Y = UV \) and \(|J| = v\).

\[ f_{X,Y}(x,y) = f_{X,Y}(y|x)f_X(x) = \frac{1}{\sqrt{2\pi x}} e^{-\frac{(y-x)^2}{2x^2}}, 0 < x < 1, -\infty < y < \infty \]
\[ f_{U,V}(u,v) = f_{X,Y}(v,uv) |J| = v \cdot \frac{1}{\sqrt{2\pi v}} e^{-\frac{(uv-v)^2}{2v^2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(u-1)^2}{2}} \]

Since \( f_{U,V}(u,v) \) is free of \( v \), \( U \) and \( V \) are independent, that is, \( Y/X \) and \( X \) are independent. The conditional distribution of \( U|X = x \) is \( n(1,1) \).

### 4.58

**a.**

\[ Cov(X,Y) = EXY - EXEY \]
\[ = E(E(XY|X)) - EXE(E(Y|X)) \]
\[ = E(XE(Y|X)) - EXE(E(Y|X)) \]
\[ = Cov(X, E(Y|X)) \]

**b.**

\[ Cov(X, Y - E(Y|X)) = Cov(X, Y) - Cov(X, E(Y|X)) = 0 \quad (by \ a) \]
c.

\[
\text{Var}(Y - E(Y|X)) = E(\text{Var}((Y - E(Y|X))|X)) + \text{Var}(E((Y - E(Y|X))|X)) \\
= E(\text{Var}(Y|X)) + \text{Var}(E(Y|X) - E(Y|X)) \\
= E(\text{Var}(Y|X))
\]

4.59

\[
E(\text{Cov}(X,Y|Z)) + \text{Cov}(E(X|Z),E(Y|Z)) \\
= E(E(XY|Z) - E(X|Z)E(Y|Z)) + E(E(X|Z)E(Y|Z)) - E(E(X|Z))E(E(Y|Z)) \\
= E(XY) - E(X|Z)E(Y|Z) + E(X|Z)E(Y|Z) - EXEY \\
= E(XY) - EXEY \\
= \text{Cov}(X,Y)
\]

4.43

\[
\text{Var}X = \begin{pmatrix}
\sigma^2 & 0 & 0 \\
0 & \sigma^2 & 0 \\
0 & 0 & \sigma^2
\end{pmatrix} = \sigma^2 I
\]

\[
\text{Cov}(X_1 + X_2, X_2 + X_3) = (1 \quad 1 \quad 0)\text{Var}X \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix} = \sigma^2
\]

\[
\text{Cov}(X_1 + X_2, X_1 - X_2) = (1 \quad 1 \quad 0)\text{Var}X \begin{pmatrix}
1 \\
-1 \\
0
\end{pmatrix} = 0
\]