# STAT542 HW2 SOLUTION

## Prob 1.6

$$p_0 = P(0\,heads\,occur) = (1-u)(1-w)$$
$$p_1 = P(1\,heads\,occur) = u(1-w) + w(1-u)$$
$$p_2 = P(2\,heads\,occur) = uw$$

Suppose $p_0 = p_1 = p_2$, $(1-u)(1-w) = u(1-w) + w(1-u) = uw$

$$\rightarrow \begin{cases} u + w = 1 \\ uw = 1/3 \end{cases}$$

These two equations imply $u(1-u) = 1/3$,which has no solution in the real numbers. Thus, the probability assignment is not legitimate.

## Prob 1.7

### a.

$$P(scoring\,i\,points) = \begin{cases} 1 - \frac{\pi r^2}{A} & i = 0 \\ \frac{\pi r^2}{A}\frac{(6-i)^2-(5-i)^2}{5^2} & i = 1,\,...,\,5 \end{cases}$$

### b.

$$P(scoring\,i\,points \cap board\,is\,hit) = \frac{\pi r^2}{A}\frac{(6-i)^2-(5-i)^2}{5^2}, \quad i = 1,\,...,\,5$$
$$P(board\,is\,hit) = \frac{\pi r^2}{A}$$

Thus

$$P(score\,i\,points \,|\, board\,i\,hit) = \frac{P(scoring\,i\,points\,\cap board\,is\,hit)}{P(board\,is\,hit)} = \frac{(6-i)^2-(5-i)^2}{5^2},$$

$i$ =1, ..., 5 which is exactly the probability distribution of Example 1.2.7.

# Prob 1.24

## a.

$$\begin{aligned} P(A\,wins) &= \sum_{i=1}^{\infty} P(A\,wins\,on\,ith\,toss) \\ &= \frac{1}{2} + (\frac{1}{2})^2\frac{1}{2} + (\frac{1}{2})^4\frac{1}{2} + ... \\ &= \sum_{i=1}^{\infty}(\frac{1}{2})^{2i+1} \\ &= \frac{1}{2}\frac{1}{1-\frac{1}{4}} \\ &= \frac{2}{3} \end{aligned}$$

## b.

$$\begin{aligned} P(A\,wins) &= \sum_{i=1}^{\infty} P(A\,wins\,on\,ith\,toss) \\ &= p + (1-p)^2p + (1-p)^4p + ... \\ &= \sum_{i=1}^{\infty}(1-p)^{2i}p \\ &= \frac{p}{1-(1-p)^2} \end{aligned}$$

## c.

From b, note that $P(A\,wins) = \frac{p}{1-(1-p)^2} \equiv f(p)$ s.t. $\frac{df(p)}{dp} = \frac{d}{dp}(\frac{p}{1-(1-p)^2}) = \frac{p^2}{(1-(1-p)^2)^2} > 0$, which means that f(p) is an increasing function in p, and the minimum value is at p=0. Using L'H öpital's rule, we could find $\lim_{p\to 0} f(p) = \lim_{p\to 0}\frac{p}{1-(1-p)^2} = \lim_{p\to 0}\frac{1}{2(1-p)} = \frac{1}{2}$. Therefore $P(A\,wins) > \frac{1}{2}$.

# Prob 1.51

The pmf of X is :

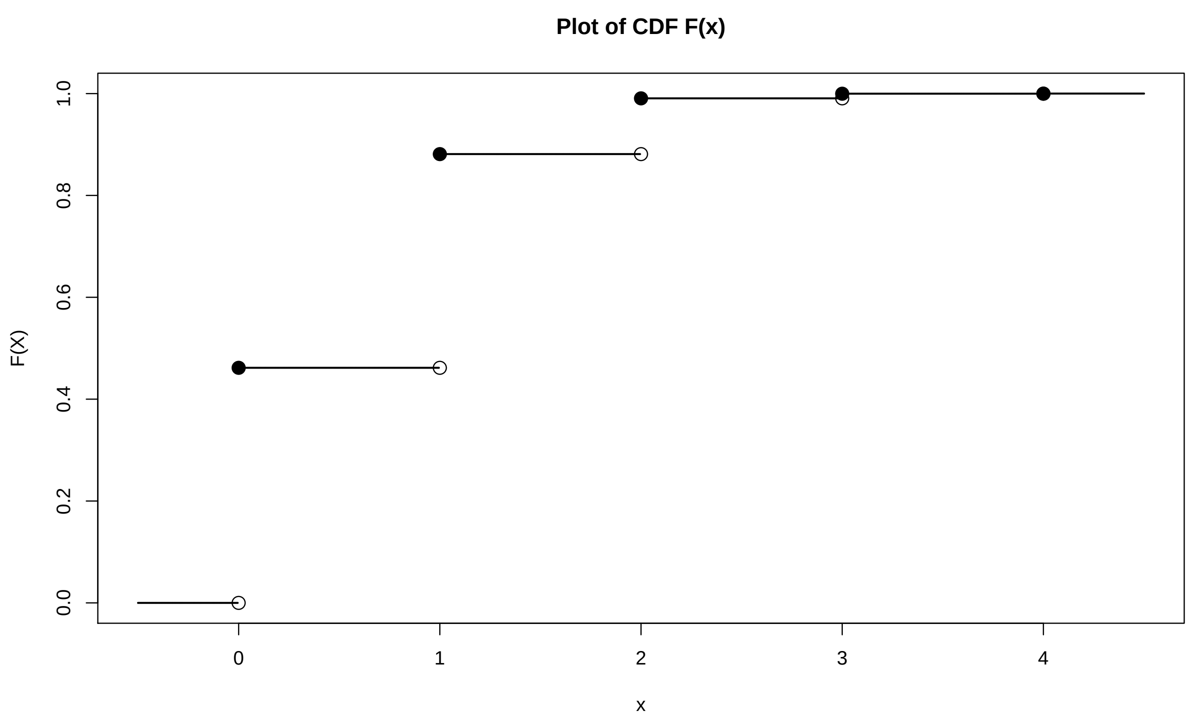
| x | $P(X = x)$ |
|---|---|
| 0 | $\binom{5}{0}\binom{25}{4}/\binom{30}{4} \approx .4616$ |
| 1 | $\binom{5}{1}\binom{25}{3}/\binom{30}{4} \approx .4196$ |
| 2 | $\binom{5}{2}\binom{25}{2}/\binom{30}{4} \approx .1095$ |

$$3 \qquad \binom{5}{3}\binom{25}{1}/\binom{30}{4} \approx .0091$$

$$4 \qquad \binom{5}{4}\binom{25}{0}/\binom{30}{4} \approx .0002$$

The cdf of X is :

$$F_X(x) = \begin{cases} 0 & -\infty < x < 0 \\ .4616 & 0 \leq x < 1 \\ .8812 & 1 \leq x < 2 \\ .9907 & 2 \leq x < 3 \\ .9998 & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$




# Prob 1.53

$$F_Y(y) = \begin{cases} 0 & y < 1 \\ 1 - \frac{1}{y^2} & y \geq 1 \end{cases}$$

## a.

i)$\lim_{y \to -\infty} F_Y(y) = \lim_{y \to -\infty} 0 = 0 \qquad \lim_{y \to \infty} F_Y(y) = \lim_{y \to \infty}(1 - \frac{1}{y^2}) = 1$

ii) For $y < 1, F_Y(y) = 0$ which is a constant function. For $y \geq 1$, $\frac{d}{dy}F_Y(y) = \frac{2}{y^3} > 0$, so $F_Y(y)$ is increasing. Thus, $F_Y(y)$ is nondecreasing for all y.

iii) For $y \neq 1, F_Y(y)$ is continuous function obviously. For y=1,$\lim_{y\to 1+} F_Y(y) = \lim_{y\to 1+}(1 - \frac{1}{y^2}) = 0 = F_Y(1)$.Thus,$F_Y(y)$ is right continuous function. Therefore, $F_Y(y)$ is cdf.

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**b.**

The pdf of Y is $f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} 0 & y < 1 \\ \frac{2}{y^3} & y \geq 1 \end{cases}$

**c.**

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(10(Y-1) \leq z) \\ &= P(Y \leq \frac{z}{10} + 1) \\ &= F_Y(\frac{z}{10} + 1) \\ &= \begin{cases} 0 & \frac{z}{10} + 1 < 1 \Rightarrow z < 0 \\ 1 - \frac{1}{(\frac{z}{10}+1)^2} & \frac{z}{10} + 1 \geq 1 \Rightarrow z \geq 0 \end{cases} \\ &= \begin{cases} 0 & z < 0 \\ \frac{z^2+20z}{(z+10)^2} & z \geq 0 \end{cases} \end{aligned}$$

# Prob 1.54

**a.**

$$1 = \int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} c\sin x\,dx = c(-cosx)|_0^{\pi/2} = c$$

Thus c=1.

**b.**

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} ce^{-|x|}\,dx = c(\int_{-\infty}^{0} e^x\,dx + \int_0^{\infty} e^{-x}\,dx) = c(e^x|_{-\infty}^0 - e^{-x}|_0^{\infty}) = 2c$$

Thus c=1/2.

# Prob 1.55

For $v < 5, P(V \leq v) = 0$

For $5 \leq v < 6$, $i.e$, $t < 3$

$$P(V \leq v) = P(V = 5) = P(0 \leq T < 3) = \int_0^3 \frac{1}{1.5}e^{-\frac{t}{1.5}}\,dt = -e^{-\frac{t}{1.5}}|_0^3 = 1 - e^{-2}$$

For $v \geq 6$, $i.e$, $t \geq 3$

$$\begin{aligned} P(V \leq v) &= P(2T \leq v) \\ &= P(T \leq v/2) \\ &= \int_0^{v/2} \frac{1}{1.5} e^{-\frac{t}{1.5}}\, dt \\ &= -e^{-\frac{t}{1.5}} |_0^{v/2} \\ &= 1 - e^{-v/3} \end{aligned}$$

$$P(V \leq v) = \begin{cases} 0 & -\infty < v < 5 \\ 1 - e^{-2} & 5 \leq v < 6 \\ 1 - e^{-v/3} & v \geq 6 \end{cases}$$