5.23

\[ P(Z > z) = \sum_{x=1}^{\infty} P(Z > z|x)P(X = x) \]
\[ = P(U_1 > z, \ldots, U_x > z|x)P(X = x) \]
\[ = \sum_{x=1}^{\infty} \prod_{i=1}^{x} P(U_i > z)P(X = x) \text{ by independence of the } U_i's \]
\[ = \sum_{x=1}^{\infty} P(U_i > z)^x P(X = x) \]
\[ = \sum_{x=1}^{\infty} (1 - z)^x \frac{1}{(e-1)x!} \]
\[ = \frac{1}{(e-1)} \sum_{x=1}^{\infty} \frac{(1-z)^x}{x!} \]
\[ = \frac{e^{1-z} - 1}{e-1}, \quad 0 < z < 1 \]

5.24

Use \( f_X(x) = 1/\theta, F_X(x) = x/\theta, 0 < x < 1. \)
Let \( Y = X_{(n)}, Z = X_{(1)}. \) Then, from Theorem 5.4.6,
\[
f_{Z,Y}(z, y) = \frac{n!}{0!(n-2)!0!\frac{1}{\theta}} \left( \frac{z}{\theta} \right)^0 \left( \frac{y-z}{\theta} \right)^{n-2} \left( 1 - \frac{y}{\theta} \right)^0 = \frac{n(n-1)}{\theta^n} (y-z)^{n-2}, \quad 0 < z < y < \theta.
\]

Now let \( W = Z/Y, Q = Y. \) Then \( Y = Q, Z = WQ, \) and \( |J| = q. \) Therefore,
\[
f_{W,Q}(w, q) = \frac{n(n-1)}{\theta^n} (q-qw)^{n-2}q = \frac{n(n-1)}{\theta^n} (1-w)^{n-2}q^{n-1}, 0 < w < 1, 0 < q < \theta.
\]
The joint pdf factors into functions of \( w \) and \( q, \) hence \( W \) and \( Q \) are independent.

5.27

a. \( f_{X(i)|X(j)}(u|v) = \frac{f_{X(i),X(j)}(u,v)}{f_{X(j)}(v)}. \) Consider tow cases, depending on which of \( i \) or \( j \) is greater.

Using the formulas from Theorem 5.4.4 and 5.4.6, and after cancellation, we obtain the following.
(i) If \( i < j \),

\[
f_{X(i|X(j))(u|v)} = \frac{(j - 1)!}{(i - 1)!(j - 1 - i)!} f_X(u) (F_X(u) - F_X(v))^j - i - 1 \left(1 - \frac{F_X(u)}{F_X(v)}\right)^j - i, \quad u < v.
\]

Note that this is the pdf of the \( i \)th order statistic from a sample of size \( j - 1 \), from a population with pdf given by the truncated distribution, \( f(u) = \frac{f_X(u)}{F_X(v)}, u < v \).

(ii) If \( i > j \) and \( u > v \),

\[
f_{X(i|X(j))(u|v)} = \frac{(n - j)!}{(n - 1)!(i - 1 - j)!} f_X(u) (1 - F_X(u))^{n - i} (F_X(u) - F_X(v))^{i - 1 - j} (1 - F_X(v))^{j - n}.
\]

This is the pdf of the \( (i-j) \)th order statistic from a sample of size \( n - j \), form a population with pdf given by the truncated distribution, \( f(u) = \frac{f_X(u)}{1 - F_X(v)}, u < v \).

### Additional Prob

1

Let \( F(X_{(n)}) = U_{(n)} \sim \text{unif}(0, 1) \), \( F(X_{(1)}) = U_{(1)} \sim \text{unif}(0, 1) \), then the joint pdf of \( U_{(n)} \) and \( U_{(1)} \) as \( f_{(1),(n)}(w, z) \) is follow:

\[
f_{(1),(n)}(w, z) = \frac{n!}{(n - 2)!} f(w)(F(z) - F(w))^{n - 2} f(z)
\]

\[
= n(n - 1)(z - w)^{n - 2}, \quad 0 < w < z < 1.
\]

\[
P((X_{(n)}) - (X_{(1)}) \geq p) = \int_0^{1-p} \int_{w+p}^1 n(n - 1)(z - w)^{n - 2} \, dz \, dw
\]

\[
= \int_0^{1-p} n(n - 1) \frac{(z - w)^{n - 1}}{n - 1} |w+p| \, dw
\]

\[
= \int_0^{1-p} n((1 - w)^{n - 1} - p^{n - 1}) \, dw
\]

\[
= -n \cdot \frac{(1 - w)^n}{n} |1 - p - np^{n - 1}(1 - p)
\]

\[
= 1 - p^n - np^{n - 1}(1 - p)
\]
a. 

\[ f(x) = k\Phi(x)\sin^2(x) \]
\[ h(x) = \Phi(x)\sin^2(x) \]
\[ g(x) = \Phi(x) \]
\[ \frac{h(x)}{g(s)} = \sin^2(x) \leq 1 \Rightarrow M = 1. \]

Then to generate \( X \) from \( kh(x) \), conduct the following steps:
1) generate \( X^* \) from \( g \).
2) generate \( U \sim \text{unif}(0, 1) \).
3) if \( U g(X^*) < h(X^*) \), then \( X = X^* \), else return to 1).

b. To approximate \( EX^2 \), we could do the following:
1) use algorithm in a) to generate a large size iid sample \( \{X_i\}_{i=1}^n \).
2) compute
\[
\frac{1}{n} \sum_{i=1}^n X_i^2 \cdot \frac{h(X_i)}{g(X_i)} = \frac{1}{n} \sum_{i=1}^n X_i^2 \sin^2(X_i) \quad (1)
\]
\[
\frac{1}{n} \sum_{i=1}^n \frac{h(X_i)}{g(X_i)} = \frac{1}{n} \sum_{i=1}^n \sin^2(X_i) \quad (2)
\]

Then \( (1)/(2) \rightarrow EX^2 = Var X \) in probability.

c. (First method)
Use empirical cdf to approximate \( P(0.3 < X < 1.2) \), that is, \( P(0.3 < X < 1.2) = \frac{1}{n} \sum_{i=1}^n I(0.3 < X_i < 1.2) \), where \( \{X_i\}_{i=1}^n \) are iid random variables.

(Second method)
Use importance sampling method to do the following:
1) use algorithm in a) to generate a large size random sample \( \{X_i\}_{i=1}^n \).
2) compute
\[
\frac{1}{n} \sum_{i=1}^n I(0.3 < X_I < 1.2) \cdot \frac{h(X_i)}{g(X_i)} = \frac{1}{n} \sum_{i=1}^n I(0.3 < X_I < 1.2)\sin^2(X_i) \quad (3)
\]
\[
\frac{1}{n} \sum_{i=1}^n \frac{h(X_i)}{g(X_i)} = \frac{1}{n} \sum_{i=1}^n \sin^2(X_i) \quad (4)
\]

Then \( (3)/(4) \rightarrow P(0.3 < X_I < 1.2) \) in probability.

3

\( p(x) \leq 1 \), let \( h(x) = I(x \in [0, 1]^5) \) be the uniform density. It could be simulated by drawing 5 iid \( \text{unif}(0,1) \) coordinates \( x_i, 1 \leq i \leq 5 \).
Note 1. $h(x) \geq p(x)$. To get a single realization $X^*$, do the following:

1) generate $X^{**} = (X_1^{**}, X_2^{**}, X_3^{**}, X_4^{**}, X_5^{**})$ from $h$.
2) generate $U \sim \text{unif}(0, 1)$ which is independent from $X^{**}$.
3) if $Uh(X^{**}) = U < p(X^{**})$, set $X^* = X^{**}$, otherwise return to 1).