Recall... independent mean squares

\[
\left( \frac{d_{i}f_{i}}{E_{i}^{MS}} \right) \sim \chi^{2}
\]

\[
S^{2} = a_{1}MS_{1} + a_{2}MS_{2} + \ldots + a_{k}MS_{k}
\]

\[
ES^{2} = a_{1}EMS_{1} + \ldots + a_{k}EMS_{k}
\]

\[
\text{Var } S^{2} = 2 \sum a_{i}^{2} \left( \frac{(EMS_{i})^{2}}{d_{i}f_{i}} \right)
\]

\[
1st \ \frac{\text{Var } S^{2}}{\text{Var } S^{2}} = 2 \sum a_{i}^{2} \left( \frac{(MS_{i})^{2}}{d_{i}f_{i}} \right)
\]

A somewhat more sophisticated estimator of \( \text{Var } S^{2} \) can be had as follows.

\[
E(\text{MS}_{i})^{2} = \text{Var } MS_{i} + (EMS_{i})^{2}
\]

\[
= \frac{2(EMS_{i})^{2}}{d_{i}f_{i}^{2}} + (EMS_{i})^{2}
\]

\[
= (EMS_{i})^{2} \cdot \frac{d_{i}f_{i}^{2} + 2}{d_{i}f_{i}^{2}}
\]

So \( (EMS_{i})^{2} = \frac{d_{i}f_{i}^{2}}{d_{i}f_{i}^{2} + 2} E(\text{MS}_{i})^{2} \) and plugging into

formula for \( \text{Var } S^{2} \) we see a 2nd possible estimator.
\[ \text{2nd} \quad \text{Var} \left( \bar{S}^2 \right) = 2 \sum i \cdot \frac{(M_{S_i})^2}{df_i} \]

This is obviously a bit smaller than the 1st estimator.

Beyond doing this kind of thing to make standard errors (est'd standard deviations) for things like \( S^2 \) it's common to make approximations to the distribution of \( S^2 \) - this is the famous Cochran-Satterthwaite approximation - approximate

\[ \frac{\nu S^2}{ES^2} \sim \chi^2 \]

where

\[ \nu = \frac{\left( ES^2 \right)^2}{\sum \frac{(x_i - EMS_i)^2}{df_i}} \]

This comes from setting \( E \frac{\nu S^2}{ES^2} = \nu \) and \( \text{Var} \frac{\nu S^2}{ES^2} = 2\nu \),

This approximation can be used to get approximate confidence intervals for \( S^2 \) - i.e.

\[ \frac{\nu S^2}{ES^2} \sim \chi^2 \]

\[ \Rightarrow \]
might work as an approximate C.I. for $\in S^2$ 
- sadly this is unavailable/ not realizable since $\nu$ is not available - but perhaps I can estimate $\nu$ with

$$\hat{\nu} = \frac{(s^2)^2}{\sum \frac{(\hat{\epsilon}_i, MS_i)^2}{d_i}}$$

and replace $\nu$ with $\hat{\nu}$ up there.

---

We will see that this kind of approximation is also sometimes used to make approximate C.I. limits for fixed effects in mixed models.

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2-Level (Balanced) Nested (Hierarchical) Models

Levels A

$\begin{array}{cccc}
1 & 2 & \ldots & b \\
1 & 2 & \ldots & b \\
1 & 2 & \ldots & b \\
\end{array}$

Level of B within A

$\begin{array}{cc}
c & \\
\end{array}$

Level of C within B within A
For example,

\[ y_{ijk} = \text{measured hardness of sample (specimen) } k \]

\[ \text{cut from inlet } j \text{ from batch } i \text{ at a steel} \]

A possible hierarchical mixed linear model for this is

\[ y_{ijk} = \mu + \alpha_i + \beta_{ij} + \epsilon_{ijk} \]

unknown constant

\[ \text{i.i.d. } N(0, \sigma^2_i), \text{ i.i.d. } N(0, \sigma^2), \text{ i.d. } N(0, \sigma^2) \]

all independent

\[ i = 1, 2, \ldots, a \quad j = 1, 2, \ldots, b \quad k = 1, 2, \ldots, c \]

This is a mixed linear model. Write out the matrix for a small case \( a = 2, b = 2, c = 2 \)

\[
Y = \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
= \begin{pmatrix}
1 \\
4
\end{pmatrix} \mu + \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\alpha_1 \\
\beta_{11} \\
\beta_{12} \\
\beta_{21} \\
\beta_{22}
\end{pmatrix} + \epsilon
\]
What is the covariance structure here?

\[ V = \mathbf{Z} \mathbf{G} \mathbf{Z}' + \sigma^2 \mathbf{I} \]

\[ \text{diag} \left( \sigma^2, \sigma^2, \sigma^2, \sigma^2, \sigma^2, \sigma^2, \sigma^2, \sigma^2 \right) \]
This can be written as

\[ V = \sigma_d^2 \mathbf{I} \otimes \mathbf{J} + \sigma^2 \mathbf{I} \otimes \mathbf{J} + \sigma^2 \mathbf{I} \]