More on the Normal version of the Gauss-Markov LM.

Testing: Consider a testable hypothesis

\[ H_0: \beta = \delta \]

in the normal Gauss-Markov model, \( Y = X\beta + \epsilon \) such that \( C = AX \) and

\[ C\hat{\beta}_{OLS} = A\hat{\gamma} \]

since \( \hat{\gamma} \) is independent of SSE, so also is \( C\hat{\beta}_{OLS} \)

and we can think of building a test of \( H_0: \beta = \delta \)

on \( C\hat{\beta}_{OLS} - \delta \) and SSE

\[ \text{MVN}(C\beta - \delta, \sigma^2 C(XX)^{-1}C') \]

Consider

\[ (C\hat{\beta}_{OLS} - \delta)' \left( \sigma^2 C(XX)^{-1}C' \right)^{-1} (C\hat{\beta}_{OLS} - \delta) \]

as a measure of apparent mismatch between the data.
(represented by $\hat{c}_L$) and the hypothesis (represented by $d$) — thus a generalization of the squared length of $T$ which $H_0$ says has mean $\bar{\omega}$ — 

Notice that

$$
\left( \sigma^2 I (X'X)^{-1} c' \right)^{-1} \left( \sigma^2 I (X'X)^{-1} c' \right) = I
$$

is idempotent, so by Koehler's Thm 4.7

$$
\left( \frac{1}{\sigma^2} (c_\beta - d)' (c_\beta - d) \right) \sim \chi^2_p (\sigma^2)
$$

for

$$
\hat{\sigma}^2 = \frac{1}{n-p} (c_\beta - d)' (c_\beta - d)
$$

In cases where $d = 0$ and we're talking about testing a hypothesis that $EY$ is in some subspace of $\langle X \rangle$ (it's not easy to show but true that this is a difference in "full model" and "reduced model" "regression sums of squares"

See "handout" for this lecture.
When \( H_0 \) is true, \( S^2 = 0 \) and when it is not true, \( S^2 > 0 \) and \( SS_{H_0} \) tends to be bigger than it would be with a noncentrality parameter - we already know that

\[
\frac{SS}{S^2} \sim X^2_{n-rank(X)}
\]

so some comparison of \( SS_{H_0} \) to \( SS/S^2 \) seems plausible as a way of testing \( H_0: \sigma^2 = \sigma^2_0 \)

**Def:** If \( U \sim \chi^2 \) independent of \( V \sim \chi^2_{\nu_2} \) then the distribution of

\[
\frac{W/\nu_1}{V/\nu_2}
\]

is called the (Snedecor) \( F \) dsn with d.f. \( \nu_1, \nu_2 \)

**Def:** If \( U \sim \chi^2_{\nu_1}(\lambda) \) independent of \( V \sim \chi^2_{\nu_2} \) then the dsn of

\[
\frac{W/\nu_1}{V/\nu_2}
\]

is called the noncentral \( F \) dsn with d.f. \( \nu_1, \nu_2 \) and noncentrality parameter \( \lambda \)
Like the $\chi^2$ dens the noncentral $F$ dens are "pulled right" relative to their central $F$ counterparts.

Define

\[ F = \frac{\frac{1}{\nu_2} \frac{\text{SS}_{H_0}}{\lambda}}{\frac{1}{\nu_1} \frac{\text{SS}_{E}/(n-\text{rank}(X))}{\text{MSE}}} = \frac{\text{MS}_{H_0}}{\text{MSE}} \]

and conclude that $F$ is $F_{(\nu_1, \nu_2)}$ with noncentrality parameter $\lambda = \frac{\nu_1}{\nu_2}$ from before.

So, an $\alpha$-level test of $H_0: \beta = 0$ can be had by rejecting $H_0$ if $F > \text{upper } \alpha \text{ pt of } F_{\nu_1, \nu_2}(0)$

or a p-value for this hypothesis is

\[
\left( \frac{\text{central } F_{\nu_1, \nu_2}}{\text{probability}} \right) \text{ to right of } \left( \frac{\text{observed value}}{\text{of } F \text{ statistic}} \right)
\]

Further I can use the non-central version of $F$ dens to to evaluate the "power" of this test - i.e. the noncentral $F$ probability to the right of (central $F$) cut-off.
\[
\text{power} = P\left(\text{test rejects } H_0\right) \\
= P\left(\frac{c_r n^{\text{ncentral}}}{\sigma^2} > \text{upper } \chi^2 \text{ pt of the central } \frac{c_r (\text{n-rank}(X))}{\text{n-rank}(X)} \text{ dsn}\right) \\
\text{for } \sigma^2 = \frac{1}{n} (c_r x - \bar{x})' (c_r (x'x) c_r')^{-1} (c_r y - \bar{y}) \\
\begin{align*}
\text{Caution} & \\
& \begin{array}{c}
\text{1 power} \\
0 \quad 0 \quad \sigma^2 \\
\end{array}
\end{align*}
\]