The following are equivalent regarding \( c \in \mathbb{R}^k \)

1) \( \exists a \in \mathbb{R}^n \) such that \( a'X\beta = c' \beta \ \forall \beta \)

2) \( c \in C(X') \) (\( c' \) is a linear combination of the rows of \( X \))

3) \( X\beta_1 = X\beta_2 \Rightarrow c'\beta_1 = c'\beta_2 \)

Condition 3) is a condition about the lack of ambiguity of the value of \( c'\beta \). Condition 1) is
\[
c'\beta = a'X\beta = a'\beta'Y = E a'Y
\]
so that \( a'Y \) can be used to estimate \( c'\beta \) in an unbiased fashion.

**Proof:** First suppose that 2) holds. \( c \in C(X') \Rightarrow \exists a \) such that \( c' = aX \). So for this \( a \),
\[
c'\beta = a'X\beta \ \forall \beta \text{ and 1) holds.}
\]

Next suppose that 1) holds, i.e. that \( \exists a \in \mathbb{R}^n \) such that \( a'X\beta = c' \beta \ \forall \beta \). Suppose \( X\beta_1 = X\beta_2 \).
For this \( a \),
\[
c'\beta_1 = a'X\beta_1 = a'X\beta_2 = c'\beta_2
\]
and 3) holds.

Finally, suppose that 3) holds. It is "obviously" equivalent to write
\[
X(\beta_1 - \beta_2) = 0 \Rightarrow c'(\beta_1 - \beta_2) = 0 \ \forall \beta_1, \beta_2
\]
That is, it is equivalent to 3) to write
\[
Xd = 0 \Rightarrow c'd = 0 \ \forall d
\]
Then, the claim that 3) \( \Rightarrow \) 2) is the claim that
\[
\{c | [Xd = 0 \Rightarrow c'd = 0 \ \forall d] \text{ holds} \} \subset C(X')
\]
Suppose that \( c \) is such that \([Xd = 0 \Rightarrow c'd = 0 \ \forall d] \) holds. Write
\[
c = P_x c + (I - P_x) c
\]
and let \( d' = (I - P_x) c \). We can then argue that \( d'^{\ast} = 0 \) as follows. Clearly, \( d' \in C(X') \) so it must be that \( c'd'^{\ast} = 0 \) from the condition \([\ ]\). But
\[
c'd'^{\ast} = c'(I - P_x) c = c'c - c'P_x c
\]
so that \( c'c = c'P_x c \). But
\[
c'c = c'(P_x + (I - P_x)) c = c'P_x c + c'(I - P_x) c = c'P_x c + c'(I - P_x)' (I - P_x) c
\]
Then since \( c'(I - P_x)' (I - P_x) c = d''d' \), we have \( d''d' = 0 \) so that \( d'^{\ast} = 0 \). Thus \( c = P_x c \) so that \( c \in C(X') \). \( \Box \)