1. Use a very small set of fake unbalanced two-level nested data.

   (a) The pooled variance from the 5 samples (of sizes 2,4,2,2, and 3) is 1.089 (standard error = 1.044). An exact 95% confidence interval for σ is $\left(\frac{1.044\sqrt{8}}{\sqrt{\chi^2_{.975,8}}}, \frac{1.044\sqrt{8}}{\sqrt{\chi^2_{.025,8}}}\right) = (0.705, 2.000)$. An approximate 95% confidence interval for σ is $(0.639, 1.747)$. The algorithm implemented in R produces a narrower confidence interval.

```
> intervals(lmef.out)
Approximate 95% confidence intervals

 Within-group standard error:
    lower      est.     upper
0.6388509 1.0565890 1.7474817
```

   (b) Note that the estimate of σ (residual standard error = 1.044) produced by `lmf.out` is exactly the one based on the mean square error referred in 1(a).

```
> summary(lmf.out)
Residual standard error: 1.044 on 8 degrees of freedom
Multiple R-Squared: 0.6286,     Adjusted R-squared: 0.4429
F-statistic: 3.385 on 4 and 8 DF,  p-value: 0.06686
```

   The predictions for the fixed effects model correspond to the cell sample means ($\bar{y}_{ij.}$)

```
> round(predict(lmf.out),3)
    1     2     3     4     5     6     7     8     9    10    11    12    13
6.050 6.050 7.400 7.400 7.400 7.400 9.650 9.650 8.500 8.500 8.067 8.067 8.067
> round(predict(lmef.out),3)
  1/1   1/1   1/2   1/2   1/2   1/2   2/1   2/1   2/2   2/2   2/3   2/3   2/3
6.739 6.739 7.210 7.210 7.210 7.210 8.881 8.881 8.530 8.530 8.354 8.354 8.354
```

2. The weights of 75 pieces of sliver were determind and a standard hierarchical (balanced data) ANOVA table was produced as below.

Term	Source	SS	df	E(MS)
A	Machines	1966	2	$\sigma^2 + 5\sigma^2_\beta + 25\sigma^2_\alpha$
B(A)	Rolls	644	12	$\sigma^2 + 5\sigma^2_\beta$
Error	Pieces	280	60	σ^2

   (a) Make 95% confidence intervals for each of the three standard deviations.

      i. A 95% confidence interval for σ is $\left(\sqrt{\frac{SSE}{\chi^2_{(.975,60)}}}, \sqrt{\frac{SSE}{\chi^2_{(.025,60)}}}\right) = (1.833, 2.630)$.

      ii. $\hat{\sigma}^2_\beta = \frac{MSB(A) - MSE}{5} = 9.8$.

      $\hat{\nu} = \frac{(\hat{\sigma}^2_\beta)^2}{\frac{(MSB(A)/5)^2}{12} + \frac{(-MSE/5)^2}{60}} = 9.989$.

      A 95% confidence interval for σ_β is $\left(\sqrt{\frac{\hat{\sigma}^2_\beta\,\hat{\nu}}{\chi^2_{(.975,\hat{\nu})}}}, \sqrt{\frac{\hat{\sigma}^2_\beta\,\hat{\nu}}{\chi^2_{(.025,\hat{\nu})}}}\right) = (2.186, 5.501)$.

      iii. $\hat{\sigma}^2_\alpha = \frac{MSA - MSB(A)}{5\times 5} = 37.173$.

      $\hat{\nu} = \frac{(\hat{\sigma}^2_\beta)^2}{\frac{(MSA/25)^2}{2} + \frac{(-MSB(A)/25)^2}{12}} = 1.787$

      A 95% confidence interval for σ_α is $\left(\sqrt{\frac{\hat{\sigma}^2_\alpha\,\hat{\nu}}{\chi^2_{(.975,\hat{\nu})}}}, \sqrt{\frac{\hat{\sigma}^2_\alpha\,\hat{\nu}}{\chi^2_{(.025,\hat{\nu})}}}\right) = (3.098, 46.298)$.

      iv. The largest part of variation in measured weight comes from differences between machines.

   (b) A 95% confidence interval for μ is $\bar{y}_{...} \pm t_{(.975,2)}\sqrt{\frac{MSA}{3\times 5\times 5}} = (19.42, 50.58)$.

3. Use fake unbalanced 3×3 factorial data of problem 4 on HW5.

```
> intervals(twoway.out)
Approximate 95% confidence intervals
 Fixed effects:
              lower     est.    upper
(Intercept) 6.74266 10.28013 13.81760    # C.I. for mu
 Random Effects:
  Level: Group
              lower     est.    upper
sd(A - 1) 0.6647270 1.875825 5.293479    # C.I. for sigma_alpha
sd(B - 1) 0.7185728 2.003537 5.586294    # C.I. for sigma_beta
 Within-group standard error:
    lower      est.     upper
0.5310667 0.8971266 1.5155087            # C.I. for sigma
```

4. Use data from page 54 of *Statistical Quality Assurance Methods for Engineers* by Vardeman and Jobe.

(a)
```
> Y <- c(13.5,14.8,10.5,11.7,12.9,12.0 ,8.8,13.5,12.4,16.0,
         11.3,12.0,14.0,12.5,13.0,13.1,12.6,12.7,11.0,10.6)
> a <- c(rep(1,10),rep(2,10))
> b <- c(rep(c(1,1,2,2,3,3,4,4,5,5),2))
> A <- as.factor(a)
> B <- as.factor(b)
> fit.out <- lm(Y~A*B)
> anova(fit.out)
Analysis of Variance Table
          Df  Sum Sq Mean Sq F value  Pr(>F)
A          1  0.5445  0.5445  0.2598 0.62129
B          4  2.6920  0.6730  0.3212 0.85751
A:B        4 24.4980  6.1245  2.9227 0.07695 .
Residuals 10 20.9550  2.0955
```

(b) A 95% confidence interval for σ is $\left(\sqrt{\frac{SSE}{\chi^2_{(.975,10)}}}, \sqrt{\frac{SSE}{\chi^2_{(.025,10)}}}\right) = (1.011, 2.540)$.

(c) $\hat{\sigma}^2_\beta + \sigma^2_{\alpha\beta} = \frac{MSB+MSAB-2MSE}{4} = .6516$.

$\hat{\nu} = \frac{(\hat{\sigma}^2_\beta+\sigma^2_{\alpha\beta})^2}{\frac{(MSB/4)^2}{4}+\frac{(MSAB/4)^2}{4}+\frac{(-MSE/2)^2}{10}} = 0.604$.

A 95% confidence interval for $\sqrt{\hat{\sigma}^2_\beta + \sigma^2_{\alpha\beta}}$ is $\left(\sqrt{\frac{\hat{\sigma}^2_\beta+\sigma^2_{\alpha\beta}\,\hat{\nu}}{\chi^2_{(.975,\hat{\nu})}}}, \sqrt{\frac{\hat{\sigma}^2_\beta+\sigma^2_{\alpha\beta}\,\hat{\nu}}{\chi^2_{(.025,\hat{\nu})}}}\right) = (0.321, 238.470)$.

(d)
```
> intervals(gauge.out)
Approximate 95% confidence intervals
 Random Effects:
  Level: Group
                   lower       est.        upper
sd(A - 1)   1.501567e-31 0.01275671 1.083759e+27
sd(B - 1)   4.061465e-22 0.01512822 5.634991e+17
sd(AB - 1)  1.260900e-01 0.70219319 3.910502e+00

 Within-group standard error:
   lower     est.    upper
0.934119 1.447563 2.243225
```

i. The point estimate of the repeatability standard deviation (σ) is 1.447. The point estimate of the reproducibility standard deviation ($\sqrt{\sigma^2_\beta + \sigma^2_{\alpha\beta}}$) is $\sqrt{0.015^2 + 0.702^2} = 0.702$ based on REML estimates.

ii. Note that the confidence interval for σ_β is extremely wide. Then, the point estimate of reproducibility standard deviation is not reliable.

iii. Our approximate interval from (c) does not do justice to the uncertainty here.

5. Use a small set of fake unbalanced two-way factorial data from 2 (unbalanced) blocks.

(a) An estimate of σ is the residual standard error (= 0.394).

```
> summary(lmf.out)
lm(formula = y ~ A * B + FBlock)
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   9.5929     0.1040  92.245 2.13e-13 ***
A1            1.2849     0.1040  12.356 1.72e-06 ***
B1           -0.2476     0.1473  -1.681    0.131
B2            1.5571     0.1541  10.104 7.86e-06 ***
FBlock1      -1.0286     0.1053  -9.768 1.01e-05 ***
A1:B1         0.2603     0.1487   1.751    0.118
A1:B2        -0.2349     0.1541  -1.524    0.166

Residual standard error: 0.394 on 8 degrees of freedom
Multiple R-Squared: 0.9793,     Adjusted R-squared: 0.9638
F-statistic: 63.07 on 6 and 8 DF,  p-value: 2.665e-06
```

(b) The inferences for the fixed effects is the same in both analyses. They give the same point estimates for the fixed effects and their standard errors.

```
> summary(lmef.out)
Linear mixed-effects model fit by REML

Random effects:
 Formula: ~1 | Block
        (Intercept)  Residual
StdDev:    1.446976 0.3940027

Fixed effects: y ~ A * B
                 Value Std.Error DF   t-value p-value
(Intercept)  9.593456 1.0284378  8  9.328183  <.0001
A1           1.284322 0.1039918  8 12.350224  <.0001
B1          -0.246421 0.1472967  8 -1.672959  0.1329
B2           1.556544 0.1541129  8 10.100021  <.0001
A1:B1        0.262713 0.1486697  8  1.767092  0.1152
A1:B2       -0.234322 0.1541129  8 -1.520455  0.1669
```

The point estimates for σ are the same in both analyses, but from `lmf.out` fitted model we obtained $\left(\frac{0.394\sqrt{8}}{\sqrt{\chi^2_{.975,8}}}, \frac{0.394\sqrt{8}}{\sqrt{\chi^2_{.025,8}}}\right) = (0.266, 0.755)$ as a 95% confidence interval for σ, and for `lmef.out` fitted model we obtained $(0.241, 0.643)$, a narrower interval.

```
> intervals(lmef.out)
Approximate 95% confidence intervals

 Random Effects:
  Level: Block
                     lower     est.    upper
sd((Intercept)) 0.3565873 1.446976 5.871606

 Within-group standard error:
    lower      est.     upper
0.2413784 0.3940027 0.6431317
```

In the fixed effects analysis the estimate of the block effect is -1.0286. In the mixed effect model τ is assumed normally distributed with mean zero and the estimate of σ_τ is 1.447. From this distribution, the value -1.0286 is within one standard deviation from its mean and, therefore consistent with the estimate of σ_τ.

(c) The BLUE of $\mu + \alpha_1 + \beta_1 + \alpha\beta_{11} + \tau_1$ is $9.592857 + 1.284920 - 0.2476190 + 0.2603175 - 1.028571 = 9.862$.
The BLUP of $\mu + \alpha_1 + \beta_1 + \alpha\beta_{11} + \tau_1$ is $9.593456 + 1.284322 - 0.2464212 + 0.2627131 - 1.017791 = 9.876$.

The BLUP of $\mu + \alpha_1 + \beta_1 + \alpha\beta_{11} + \tau_3$ is $9.593456 + 1.284322 - 0.2464212 + 0.2627131 + 0 = 10.894$.
In the fixed effects model we require information from the third block in order to estimate its effect and to predict a new observation from this block.

```
> random.effects(lmef.out)
  (Intercept)
1   -1.017791
2    1.017791
```

6. Consider the scenario of the example beginning on page 1190 of Neter, Wasserman, and friends.

(a)
```
> anova(lmf1.out)
Analysis of Variance Table

Response: y
             Df  Sum Sq Mean Sq F value Pr(>F)
time          2   67073   33537                  # SSA    =   67073
ad            1  168151  168151                  # SSB    =  168151
time:ad       2     391     196                  # SSAB   =     391
ad:test       8 1833681  229210                  # SSC(B) = 1833681
time:ad:test 16    5727     358                  # SSE    =    5727
Residuals     0       0

> anova(lmf2.out)
Analysis of Variance Table

Response: y
             Df  Sum Sq Mean Sq F value Pr(>F)
time          2   67073   33537                  # SSA    =   67073
ad            1  168151  168151                  # SSB    =  168151
test          4  397490   99373
time:ad       2     391     196                  # SSAB   =     391
time:test     8    2475     309
ad:test       4 1436191  359048
time:ad:test  8    3252     407
Residuals     0       0
```

SSC(B) = SS(test) + SS(ad:test) and Df(C(B)) = Df(test) + Df(ad:test).
SSE = SS(time:test) + SS(time:ad:test) and Df(Error) = Df(time:test) + Df(time:ad:test).

(b) An approximate 95% confidence interval for σ_γ is $(169.076, 451.181)$ and an approximate 95% confidence interval for σ is $(13.380, 26.754)$.

```
> intervals(lmef.out)
Approximate 95% confidence intervals
 Random Effects:
  Level: loc
                    lower     est.    upper
sd((Intercept)) 169.0764 276.1957 451.1809

 Within-group standard error:
   lower     est.    upper
13.37975 18.92001 26.75436
```

(c) Exact 95% confidence interval for σ is $\left(\sqrt{\frac{5727}{\chi^2_{.975,16}}}, \sqrt{\frac{5727}{\chi^2_{.025,16}}}\right) = (14.090, 28.794)$.

$\hat{\sigma}^2_\gamma = \frac{MSC(B)-MSE}{3} = 229210 - 3583 = 76284$ and, $\hat{\nu} = \frac{76284^2}{\frac{((1/3)229210)^2}{8} + \frac{((-1/3)358)^2}{16}} = 7.975$. Therefore, an approximate 95% confidence interval for σ_γ is $\left(\sqrt{\frac{76284\hat{\nu}}{\chi^2_{.975,\hat{\nu}}}}, \sqrt{\frac{76284\hat{\nu}}{\chi^2_{.025,\hat{\nu}}}}\right) = (186.467, 529.839)$.

These intervals are wider than those obtained in (b).

(d) A 95% confidence interval for the difference in Time 1 and Time 2 main effects is $(\bar{y}_{1..} - \bar{y}_{2..}) \pm t_{.975,16}\sqrt{\frac{2}{10}MSE} = (-98.4, -62.5)$. A 95% confidence interval for the difference in Ad Campaign 1 and Ad Campaign 2 main effects is $(\bar{y}_{.1.} - \bar{y}_{.2.}) \pm t_{.975,8}\sqrt{\frac{2}{15}MSC(B)} = (-253.4, 552.9)$. This last interval can also be obtained using `intervals(lmef.out)` and, doubling the limits of the interval for the fixed effect `ad1`.

```
> intervals(lmef.out)
Approximate 95% confidence intervals

 Fixed effects:
                  lower        est.      upper
(Intercept)  479.234485 664.5333333 849.832182
time1        -26.489335 -16.1333333  -5.777331
time2         53.910665  64.2666667  74.622669
ad1         -126.698897  74.8666667 276.432230
time1:ad1    -15.022669  -4.6666667   5.689335
time2:ad1     -9.822669   0.5333333  10.889335
```

(e) We can predict $Y_{22k} = 664.5333 + 64.2667 - 74.8667 - 0.5333 + 0 = 653.4$ for $k = new$.

```
> summary(lmef.out)
Fixed effects: y ~ 1 + time * ad
               Value Std.Error DF   t-value p-value
(Intercept) 664.5333  87.40902 16  7.602571  <.0001
time1       -16.1333   4.88512 16 -3.302543  0.0045
time2        64.2667   4.88512 16 13.155583  <.0001
ad1          74.8667  87.40902  8  0.856510  0.4166
time1:ad1    -4.6667   4.88512 16 -0.955281  0.3536
time2:ad1     0.5333   4.88512 16  0.109175  0.9144
```