1. $\mu+\tau_1$, $2\mu+\tau_1+\tau_2$, $\tau_1-\tau_2$ and $(\tau_1-\tau_2)-(\tau_3-\tau_4)$ are estimable.

   (a) Let $\underline{c}' = (0,1,0,0,0)$. $P_{X'}\underline{c} \neq \underline{c}$, then τ_1 is not estimable.

   (b) Let $\underline{c}' = (1,1,0,0,0)$. $P_{X'}\underline{c} = \underline{c}$, then $\mu+\tau_1$ is estimable.
   - $\underline{c}'(X'X)^{-}X' = (1/2, 1/2, 0, 0, 0, 0)$.

   (c) Let $\underline{c}' = (1,1,1,0,0)$. $P_{X'}\underline{c} \neq \underline{c}$, then $\mu+\tau_1+\tau_2$ is not estimable.

   (d) Let $\underline{c}' = (2,1,1,0,0)$. $P_{X'}\underline{c} = \underline{c}$, then $2\mu+\tau_1+\tau_2$ is estimable.
   - $\underline{c}'(X'X)^{-}X' = (1/2, 1/2, 1, 0, 0, 0)$.

   (e) Let $\underline{c}' = (0,1,-1,0,0)$. $P_{X'}\underline{c} = \underline{c}$, then $\tau_1-\tau_2$ is estimable.
   - $\underline{c}'(X'X)^{-}X' = (1/2, 1/2, -1, 0, 0, 0)$.

   (f) Let $\underline{c}' = (0,1,-1,-1,1)$. $P_{X'}\underline{c} = \underline{c}$, then $(\tau_1-\tau_2)-(\tau_3-\tau_4)$ is estimable.
   - $\underline{c}'(X'X)^{-}X' = (1/2, 1/2, -1, -1, 1/2, 1/2)$.

2.
```
> library(MASS)
> project <- function(A) {A%*%ginv(t(A)%*%A)%*%t(A)}
> X <- matrix(c(rep(1,8),rep(c(rep(0,6),1),3),1),6,5)
> round(project(X),1)                      > round(project(X)%*%X)
     [,1] [,2] [,3] [,4] [,5] [,6]              [,1] [,2] [,3] [,4] [,5]
[1,]  0.5  0.5    0    0  0.0  0.0         [1,]    1    1    0    0    0
[2,]  0.5  0.5    0    0  0.0  0.0         [2,]    1    1    0    0    0
[3,]  0.0  0.0    1    0  0.0  0.0         [3,]    1    0    1    0    0
[4,]  0.0  0.0    0    1  0.0  0.0         [4,]    1    0    0    1    0
[5,]  0.0  0.0    0    0  0.5  0.5         [5,]    1    0    0    0    1
[6,]  0.0  0.0    0    0  0.5  0.5         [6,]    1    0    0    0    1

> round(project(t(X)),1)                   > round(project(t(X))%*%t(X))
     [,1] [,2] [,3] [,4] [,5]                   [,1] [,2] [,3] [,4] [,5] [,6]
[1,]  0.8  0.2  0.2  0.2  0.2              [1,]    1    1    1    1    1    1
[2,]  0.2  0.8 -0.2 -0.2 -0.2              [2,]    1    1    0    0    0    0
[3,]  0.2 -0.2  0.8 -0.2 -0.2              [3,]    0    0    1    0    0    0
[4,]  0.2 -0.2 -0.2  0.8 -0.2              [4,]    0    0    0    1    0    0
[5,]  0.2 -0.2 -0.2 -0.2  0.8              [5,]    0    0    0    0    1    1
```

3. $\mu+\alpha_1+\beta_1+\alpha\beta_{11}$ and $(\alpha\beta_{12}-\alpha\beta_{11})-(\alpha\beta_{22}-\alpha\beta_{21})$ are estimable.

   (a) Let $\underline{c}' = (0,1,0,0,0,0,0,0,0)$. $P_{X'}\underline{c} \neq \underline{c}$, then α_1 is not estimable.

   (b) Let $\underline{c}' = (0,1,-1,0,0,0,0,0,0)$. $P_{X'}\underline{c} \neq \underline{c}$, then $\alpha_1-\alpha_2$ is not estimable.

   (c) Let $\underline{c}' = (1,1,0,1,0,0,0,0,0)$. $P_{X'}\underline{c} \neq \underline{c}$, then $\mu+\alpha_1+\beta_1$ is not estimable.

   (d) Let $\underline{c}' = (1,1,0,1,0,1,0,0,0)$. $P_{X'}\underline{c} = \underline{c}$, then $\mu+\alpha_1+\beta_1+\alpha\beta_{11}$ is estimable.
   - $\underline{c}'(X'X)^{-}X' = (1/2, 1/2, 0, 0, 0, 0, 0, 0)$.

   (e) Let $\underline{c}' = (0,0,0,0,0,1,0,0,0)$. $P_{X'}\underline{c} \neq \underline{c}$, then $\alpha\beta_{11}$ is not estimable.

   (f) Let $\underline{c}' = (0,0,0,0,0,-1,1,1,-1)$. $P_{X'}\underline{c} = \underline{c}$, then $(\alpha\beta_{12}-\alpha\beta_{11})-(\alpha\beta_{22}-\alpha\beta_{21})$ is estimable.
   - $\underline{c}'(X'X)^{-}X' = (-1/2, -1/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2)$.

   (g) $Rank(C) = 2 = l$, but $\underline{c}_1'\beta$ is not estimable (from 3(b)). Therefore, $H_0: C\beta = 0$ is not testable.

4. $H_0: C\beta = d$ can be written as $(0, x-x', x^2-x'^2)\begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\end{bmatrix} = 0$

5. Note that $E(Y) = X\beta$ and $Var(Y) = \sigma^2 I$.

   (a) Find $E(Y - \widehat{Y})$ and $Var(Y - \widehat{Y})$.

      i. $E(Y - \widehat{Y}) = E[(I - P_X)Y] = (I - P_X)E(Y) = (I - P_X)X\beta = (X - P_X X)\beta = 0$.

      ii. $Var(Y - \widehat{Y}) = Var[(I - P_X)Y] = (I - P_X)Var(Y)(I - P_X)' = (I - P_X)\sigma^2 I(I - P_X)' = \sigma^2(I - P_X)$.

   (b) Show that every entry of $Y - \widehat{Y}$ is uncorrelated with every entry of $\widehat{Y}$. (i.e., show that $cov(\widehat{Y}, Y - \widehat{Y}) = 0$.
   $cov(\widehat{Y}, Y - \widehat{Y}) = cov[P_X Y, (I - P_X)Y] = P_X Var(Y)(I - P_X)' = \sigma^2 P_X(I - P_X)' = \sigma^2(P_X - P_X) = 0$.

   (c) Show that $E(Y - \widehat{Y})'(Y - \widehat{Y}) = \sigma^2(n - rank(X))$.

$$
\begin{aligned}
E(Y - \widehat{Y})'(Y - \widehat{Y}) &= E[((I - P_X)Y)'(I - P_X)Y] = E[Y'(I - P_X)Y] \\
&= tr[(I - P_X)\sigma^2 I] + (X\beta)'(I - P_X)X\beta \\
&= \sigma^2 tr(I - P_X) + (X\beta)'(X - P_X X)\beta \\
&= \sigma^2[tr(I) - tr(P_X)] = \sigma^2[n - rank(X)]
\end{aligned}
$$

6.
```
library(MASS)
project <- function(A) {A%*%ginv(t(A)%*%A)%*%t(A)}
inv.half <- function(A) {EV <- eigen(A); EV$vectors%*%diag(1/sqrt(EV$values))%*%t(EV$vectors)}
half <- function(A) {EV <- eigen(A); EV$vectors%*%diag(sqrt(EV$values))%*%t(EV$vectors)}

X <- matrix(c(rep(1,8),rep(c(rep(0,6),1),3),1),6,5)
Y <- c(2,1,4,6,3,5)
C <- matrix(c(rep(1,5),rep(c(rep(0,4),1),3)),4,5)
V1 <- diag(c(1,4,4,1,1,4))
V2 <- diag(c(1,4,4,1,1,4))
V2[1,2]<-1; V2[2,1]<-1; V2[3,4]<- -1; V2[4,3]<- -1; V2[5,6] <- -1; V2[6,5]<- -1

V <- V1                             # Or V <- V2
U <- inv.half(V)%*%Y
W <- inv.half(V)%*%X
OLS.U <- ginv(t(W)%*%W)%*%t(W)%*%U
Uhat <- W%*%OLS.U
Yhatstar <- half(V)%*%Uhat          # Estimate of E(Y)
Cbetahat <- C%*%OLS.U               # Estimate of Cbeta
round(X%*%ginv(t(W)%*%W)%*%t(X),4)   # cov Yhatstar for GLS estimators
round(project(X)%*%V%*%project(X),4) # cov Yhatstar for OLS estimators
round(C%*%ginv(t(W)%*%W)%*%t(C),4)   # cov Cbetahat for GLS estimators
round(C%*%ginv(t(X)%*%X)%*%t(X)%*%V%*%X%*%ginv(t(X)%*%X)%*%t(C),4) # cov Cbetahat for OLS estimators
```

With V_1:

$$
\widehat{Y} = \begin{bmatrix} 1.8 \\ 1.8 \\ 4.0 \\ 6.0 \\ 3.4 \\ 3.4 \end{bmatrix}
Cov(\widehat{Y})_{GLS} = \begin{bmatrix} 0.8 & 0.8 & 0 & 0 & 0.0 & 0.0 \\ 0.8 & 0.8 & 0 & 0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 4 & 0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0 & 1 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0 & 0 & 0.8 & 0.8 \\ 0.0 & 0.0 & 0 & 0 & 0.8 & 0.8 \end{bmatrix}
Cov(\widehat{Y})_{OLS} = \begin{bmatrix} 1.25 & 1.25 & 0 & 0 & 0.00 & 0.00 \\ 1.25 & 1.25 & 0 & 0 & 0.00 & 0.00 \\ 0.00 & 0.00 & 4 & 0 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0 & 1 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0 & 0 & 1.25 & 1.25 \\ 0.00 & 0.00 & 0 & 0 & 1.25 & 1.25 \end{bmatrix}.
$$

$$
\widehat{C\beta} = \begin{bmatrix} 1.8 \\ 4.0 \\ 6.0 \\ 3.4 \end{bmatrix}
Cov(\widehat{C\beta})_{GLS} = \begin{bmatrix} 0.8 & 0 & 0 & 0.0 \\ 0.0 & 4 & 0 & 0.0 \\ 0.0 & 0 & 1 & 0.0 \\ 0.0 & 0 & 0 & 0.8 \end{bmatrix}
Cov(\widehat{C\beta})_{OLS} = \begin{bmatrix} 1.25 & 0 & 0 & 0.00 \\ 0.00 & 4 & 0 & 0.00 \\ 0.00 & 0 & 1 & 0.00 \\ 0.00 & 0 & 0 & 1.25 \end{bmatrix}.
$$

With V_2:

$$\widehat{Y} = \begin{bmatrix} 2.0000 \\ 2.0000 \\ 4.0000 \\ 6.0000 \\ 3.5714 \\ 3.5714 \end{bmatrix} \quad Cov(\widehat{Y})_{GLS} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0.0000 & 0.0000 \\ 1 & 1 & 0 & 0 & 0.0000 & 0.0000 \\ 0 & 0 & 4 & -1 & 0.0000 & 0.0000 \\ 0 & 0 & -1 & 1 & 0.0000 & 0.0000 \\ 0 & 0 & 0 & 0 & 0.4286 & 0.4286 \\ 0 & 0 & 0 & 0 & 0.4286 & 0.4286 \end{bmatrix} \quad Cov(\widehat{Y})_{OLS} = \begin{bmatrix} 1.75 & 1.75 & 0 & 0 & 0.00 & 0.00 \\ 1.75 & 1.75 & 0 & 0 & 0.00 & 0.00 \\ 0.00 & 0.00 & 4 & -1 & 0.00 & 0.00 \\ 0.00 & 0.00 & -1 & 1 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0 & 0 & 0.75 & 0.75 \\ 0.00 & 0.00 & 0 & 0 & 0.75 & 0.75 \end{bmatrix}$$

$$\widehat{C\beta} = \begin{bmatrix} 2.0000 \\ 4.0000 \\ 6.0000 \\ 3.5714 \end{bmatrix} \quad Cov(\widehat{C\beta})_{GLS} = \begin{bmatrix} 1 & 0 & 0 & 0.0000 \\ 0 & 4 & -1 & 0.0000 \\ 0 & -1 & 1 & 0.0000 \\ 0 & 0 & 0 & 0.4286 \end{bmatrix} \quad Cov(\widehat{C\beta})_{OLS} = \begin{bmatrix} 1.75 & 0 & 0 & 0.00 \\ 0.00 & 4 & -1 & 0.00 \\ 0.00 & -1 & 1 & 0.00 \\ 0.00 & 0 & 0 & 0.75 \end{bmatrix}.$$

Covariance matrices of GLS estimators are smaller than covariances of OLS estimators.

7. $H_0 : C\beta = 0$

(a)
```
> A
     [,1] [,2]  [,3]  [,4]  [,5]  [,6]  [,7]  [,8]
[1,] 0.50 0.50 -0.50 -0.50 -0.50 -0.50  0.50  0.50
[2,] 0.25 0.25  0.25  0.25 -0.25 -0.25 -0.25 -0.25
```

(b) (From Handwritten Notes p.14)

- The model $Y = X\beta + \underline{\varepsilon}$ says $E(Y) \in C(X)$.
- The hypothesis $H_0 : C\beta = AX\beta = AE(Y) = 0$ means that $E(Y) = X\beta$ is orthogonal to each row of A, i.e., $E(Y) \in C(A')^{\perp}$ and $C(X) \cap C(A')^{\perp}$ is a subspace of $C(X)$ of dimension smaller than $dim[C(X)] = rank(X)$, namely $rank(X) - rank(A')$.
- For the proposed X_0, $C(X_0) \subset C(X)$.
- Using R we show that $P_{A'}X_0 = 0$. Thus, $C(X_0) \cap C(A')^{\perp}$.
- $rank(X_0) = 2 = rank(X) - rank(A')$.

Therefore, $H_0 : C\beta = 0$ can be written as $H_0 : E(Y) \in C(X_0)$.

```
> X0 <- X[,c(1,4,5)]
> project(t(A))%*%X0
     [,1] [,2] [,3]
[1,]    0    0    0
[2,]    0    0    0
[3,]    0    0    0
[4,]    0    0    0
[5,]    0    0    0
[6,]    0    0    0
[7,]    0    0    0
[8,]    0    0    0
> qr(X0)$rank
[1] 2
> qr(X)$rank
[1] 4
> qr(t(A))$rank
[1] 2
```

(c) $H_0 : C\beta = 0$ tests the hypothesis of "B main effects only".