1. Write out the models in the matrix form $Y = X\beta + \varepsilon$.

(a) $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_5^2 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix}.$$

(b) $$\begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & x_{111} - \bar{x}_{...} \\ 1 & 1 & 0 & 1 & 0 & x_{112} - \bar{x}_{...} \\ 1 & 1 & 0 & 0 & 1 & x_{121} - \bar{x}_{...} \\ 1 & 1 & 0 & 0 & 1 & x_{122} - \bar{x}_{...} \\ 1 & 0 & 1 & 1 & 0 & x_{211} - \bar{x}_{...} \\ 1 & 0 & 1 & 1 & 0 & x_{212} - \bar{x}_{...} \\ 1 & 0 & 1 & 0 & 1 & x_{221} - \bar{x}_{...} \\ 1 & 0 & 1 & 0 & 1 & x_{222} - \bar{x}_{...} \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \gamma \end{bmatrix} + \begin{bmatrix} \varepsilon_{111} \\ \varepsilon_{112} \\ \varepsilon_{121} \\ \varepsilon_{122} \\ \varepsilon_{211} \\ \varepsilon_{212} \\ \varepsilon_{221} \\ \varepsilon_{222} \end{bmatrix}.$$

2. Remember that generalized inverses are not unique.

```
> A <- matrix(c(4,rep(2,4),rep(c(0,2),2)),3,3,byrow=T)
> G1 <- matrix(c(rep(0,4),.5,0,0,0,.5),3,3,byrow=T)
> G2 <- matrix(c(0,0,0,.5,.0,-.5,0,0,.5),3,3,byrow=T)
> G1                       > G2                        > A%*%G1%*%A          > A%*%G2%*%A
     [,1] [,2] [,3]             [,1] [,2] [,3]              [,1] [,2] [,3]        [,1] [,2] [,3]
[1,]    0  0.0  0.0        [1,]  0.0    0  0.0         [1,]    4    2    2   [1,]    4    2    2
[2,]    0  0.5  0.0        [2,]  0.5    0 -0.5         [2,]    2    2    0   [2,]    2    2    0
[3,]    0  0.0  0.5        [3,]  0.0    0  0.5         [3,]    2    0    2   [3,]    2    0    2
```

The function `ginv()` did not return either of the generalized inverses I found "by hand".

```
> ginv(A)                                        > round(A%*%ginv(A)%*%A,0)
           [,1]        [,2]        [,3]               [,1] [,2] [,3]
[1,] 0.11111111  0.05555556  0.05555556          [1,]    4    2    2
[2,] 0.05555556  0.27777778 -0.22222222          [2,]    2    2    0
[3,] 0.05555556 -0.22222222  0.27777778          [3,]    2    0    2
```

3. Show that for M is the projection operator onto $C(X)$, where

$$X = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & .5 \\ 0 & 0 & 0 & 0 & .5 & .5 \end{bmatrix}$$

i. $C(X)$ is the set of all linear combinations of the columns of X.

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a_1 + a_2 \\ a_1 + a_2 \\ a_1 + a_3 \\ a_1 + a_4 \\ a_1 + a_5 \\ a_1 + a_5 \end{bmatrix} = \begin{bmatrix} a \\ a \\ b \\ c \\ d \\ d \end{bmatrix}$$

Similarly, $C(M)$ also consists of vectors of the form $(a, a, b, c, d, d)'$. Using R, we verify that M is symmetric ($M' = M$) and idempotent ($MM = M$). Therefore, by Theorem B.33, M is a perpendicular projection operator on $C(M)$, and since $C(M) = C(X)$, M is a perpendicular projection operator on $C(X)$.

```
> X <- matrix(c(rep(1,8),rep(c(rep(0,6),1),3),1),6,5)
> M <- matrix(c(rep(c(.5,.5,0,0,0,0),2),0,0,1,rep(0,6),1,rep(0,6),.5,.5,rep(0,4),.5,.5),6,6,byrow=
> t(M)                                       > M%*%M
     [,1] [,2] [,3] [,4] [,5] [,6]                [,1] [,2] [,3] [,4] [,5] [,6]
[1,]  0.5  0.5    0    0  0.0  0.0           [1,]  0.5  0.5    0    0  0.0  0.0
[2,]  0.5  0.5    0    0  0.0  0.0           [2,]  0.5  0.5    0    0  0.0  0.0
[3,]  0.0  0.0    1    0  0.0  0.0           [3,]  0.0  0.0    1    0  0.0  0.0
[4,]  0.0  0.0    0    1  0.0  0.0           [4,]  0.0  0.0    0    1  0.0  0.0
[5,]  0.0  0.0    0    0  0.5  0.5           [5,]  0.0  0.0    0    0  0.5  0.5
[6,]  0.0  0.0    0    0  0.5  0.5           [6,]  0.0  0.0    0    0  0.5  0.5
```

⋆i. [Another choice to show $C(X) = C(M)$]. Let $X = (\underline{X}_1, \ldots, \underline{X}_5)$ and $M = (\underline{M}_1, \ldots, \underline{M}_6)$. Note that every column in X can be written as a linear combination of columns in M. Therefore, $C(X) = C(M)$.
$X_1 = \sum_{i=1}^{5} M_i$, $X_2 = M_1 + M_2$, $X_3 = M_3$, $X_4 = M_4$, $X_5 = M_5 + M_6$ and
$M_1 = \frac{1}{2}X_2$, $M_2 = \frac{1}{2}X_2$, $M_3 = X_3$, $M_4 = X_4$, $M_5 = \frac{1}{2}X_5$, $M_6 = \frac{1}{2}X_5$.

ii. We compute $P_X = X(X'X)^{-}X'$ which is the projection operator onto $C(X)$ and, it matches the matrix proposed in the statement of this question.

```
> Px <- X%*%ginv(t(X)%*%X)%*%t(X)
> round(Px,1)
     [,1] [,2] [,3] [,4] [,5] [,6]
[1,]  0.5  0.5    0    0  0.0  0.0
[2,]  0.5  0.5    0    0  0.0  0.0
[3,]  0.0  0.0    1    0  0.0  0.0
[4,]  0.0  0.0    0    1  0.0  0.0
[5,]  0.0  0.0    0    0  0.5  0.5
[6,]  0.0  0.0    0    0  0.5  0.5
```

4. For the same X as in question 3, suppose that $Y = (2, 1, 4, 6, 3, 5)'$.

(a) Let G_1 and G_2 be two different generalized inverses for $(X'X)$, and let $\underline{b_i} = G_i X'Y$ for $i = 1, 2$.

$$G_1 = \begin{bmatrix} 0 & 0.0 & 0 & 0 & 0.0 \\ 0 & 0.5 & 0 & 0 & 0.0 \\ 0 & 0.0 & 1 & 0 & 0.0 \\ 0 & 0.0 & 0 & 1 & 0.0 \\ 0 & 0.0 & 0 & 0 & 0.5 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.12 & -0.02 & 0.08 & 0.08 & -0.02 \\ -0.02 & 0.42 & -0.18 & -0.18 & -0.08 \\ 0.08 & -0.18 & 0.72 & -0.28 & -0.18 \\ 0.08 & -0.18 & -0.28 & 0.72 & -0.18 \\ -0.02 & -0.08 & -0.18 & -0.18 & 0.42 \end{bmatrix}.$$

Note that $\hat{Y} = X\underline{b_i} = (1.5, 1.5, 4, 6, 4, 4)'$ for $i = 1, 2$; but $\underline{b}_1 = (0, .5, 4, 6, 4)'$ and $\underline{b}_2 = (3.1, -1.6, 0.9, 2.9, 0.9)'$.

(b)
```
> X <- matrix(c(rep(1,8),rep(c(rep(0,6),1),3),1),6,5)
> Y <- c(2,1,4,6,3,5)
> Px <- X%*%ginv(t(X)%*%X)%*%t(X)
> Yhat <- Px%*%Y
```

i. $\hat{Y} = (1.5, 1.5, 4, 6, 4, 4)'$.

ii. $Y - \hat{Y} = (0.5, -0.5, 0, 0, -1, 1)'$.

iii. $\hat{Y}'(Y - \hat{Y}) = 0$.

iv. $Y'Y = 91$.

v. $\hat{Y}'\hat{Y} = 88.5$.

vi. $(Y - \hat{Y})'(Y - \hat{Y}) = 2.5$.

5. Let $Y = (y_1, y_2, y_3)'$ with $Y\ N(\mu, V)$ where $\mu = (0, 1, 0)'$ and $V = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 10 \end{bmatrix}$.

(a) $y_3 \sim N(0, 10)$.

(b) $\begin{bmatrix} y_1 \\ y_3 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ -1 & 10 \end{bmatrix} \right)$.

(c) $y_3 | y_1 = 2 \sim N(-2/3, 29/3)$.

(d) $$y_3|y_1=2,y_2=-1 \sim N\left(0+\begin{bmatrix}-1 & 0\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 5\end{bmatrix}^{-1}\begin{bmatrix}2-0\\ -1-1\end{bmatrix}, 10-\begin{bmatrix}-1 & 0\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 5\end{bmatrix}^{-1}\begin{bmatrix}-1\\ 0\end{bmatrix}\right)$$
$$= N(-2/3, 29/3) \quad f(y_3|y_1,y_2)=f(y_3|y_1).$$

(e) $$\begin{bmatrix}y_1\\ y_3\end{bmatrix}\Bigg| y_2=-1 \sim N\left(\begin{bmatrix}0\\ 0\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}[1/5][-1-1], \begin{bmatrix}3 & -1\\ -1 & 10\end{bmatrix}-\begin{bmatrix}0\\ 0\end{bmatrix}[1/5]\begin{bmatrix}0 & 0\end{bmatrix}\right)$$
$$= N\left(\begin{bmatrix}0\\ 0\end{bmatrix}, \begin{bmatrix}3 & -1\\ -1 & 10\end{bmatrix}\right) \quad f(y_1,y_3|y_2)=f(y_1,y_3).$$

(f) Find the correlations ρ_{12}, ρ_{13}, ρ_{23}

i. $\rho_{12}=0/\sqrt{3(5)}=0$.

ii. $\rho_{13}=-1/\sqrt{3(10)}=-1/\sqrt{30}$.

iii. $\rho_{23}=0/\sqrt{5(10)}=0$.

(g) $$Z \sim N\left(\begin{bmatrix}1 & -1 & 1\\ 3 & 1 & 0\end{bmatrix}E(Y)+\begin{bmatrix}0\\ 1\end{bmatrix}, \begin{bmatrix}1 & -1 & 1\\ 3 & 1 & 0\end{bmatrix}Cov(Y)\begin{bmatrix}1 & 3\\ -1 & 1\\ 1 & 0\end{bmatrix}\right)=N\left(\begin{bmatrix}-1\\ 2\end{bmatrix}, \begin{bmatrix}16 & 1\\ 1 & 32\end{bmatrix}\right).$$

6. Using the function `eigen()` to obtain eigenvalues and eigenvectors of a matrix V we are able to compute its inverse square root, $W=V^{-1/2}=UD^{-1/2}U'$. Note that $WW=V^{-1}$.

```
> V <- matrix(c(3,-1,1,-1,5,-1,1,-1,3),3,3,byrow=T)
> EV <- eigen(V)
> W <- EV$vectors%*%diag(1/sqrt(EV$values))%*%t(EV$vectors)
> W
            [,1]       [,2]        [,3]
[1,]  0.61404486 0.05636733 -0.09306192
[2,]  0.05636733 0.46461562  0.05636733
[3,] -0.09306192 0.05636733  0.61404486
> W%*%W
            [,1]       [,2]        [,3]
[1,]  0.38888889 0.05555556 -0.11111111
[2,]  0.05555556 0.22222222  0.05555556
[3,] -0.11111111 0.05555556  0.38888889
> solve(V)
            [,1]       [,2]        [,3]
[1,]  0.38888889 0.05555556 -0.11111111
[2,]  0.05555556 0.22222222  0.05555556
[3,] -0.11111111 0.05555556  0.38888889
```

7.
```
> A <- matrix(c(4,4.001,4.001,4.002),2,2)
> B <- matrix(c(4,4.001,4.001,4.002001),2,2)
> det(A);det(B)         # determinants of A and B, respectively.
[1] -1e-06          # 4*4.002-4.001*4.001
[1] 3e-06           # 4*4.002001-4.001*4.001
> ginv(A)
         [,1]     [,2]
[1,] -4002000  4001000
[2,]  4001000 -4000000
> ginv(B)
         [,1]     [,2]
[1,]  1334000 -1333667
[2,] -1333667  1333333
> 3*ginv(B)
         [,1]     [,2]
[1,]  4002001 -4001000
[2,] -4001000  4000000
```