

Stat 511 HW#1 Spring 2008

1. Write out the following models of elementary/intermediate statistical analysis in the matrix form $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$:

a) A special two-variable quadratic polynomial regression model

$$y_i = \alpha_0 + \alpha_1 x_{1i} + \alpha_2 x_{1i}^2 + \alpha_3 x_{2i} + \varepsilon_i$$

for $i = 1, 2, \dots, 5$

b) A two-factor ANOVA model without interactions

$$y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}$$

for $i = 1, 2, j = 1, 2, \text{ and } k = 1, 2.$

2. Find two different generalized inverses of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

See, for example, panels 162-168 of Koehler's notes or use the following algorithm (from a book by Searle): To find a generalized inverse for an $n \times k$ matrix \mathbf{A} of rank r :

1. Find any non-singular $r \times r$ sub-matrix of \mathbf{A} , \mathbf{C} . (It is not necessary that the elements of \mathbf{C} occupy adjacent rows or columns of \mathbf{A} .)
2. Find \mathbf{C}^{-1} and $(\mathbf{C}^{-1})'$.
3. Replace the elements of \mathbf{C} with the elements of $(\mathbf{C}^{-1})'$.
4. Replace all other elements of \mathbf{A} with 0's.
5. Transpose the resulting matrix.

Then load the MASS package in R and use the function `ginv()` provided by that package to find a generalized inverse. Does R return either of the generalized inverses you found "by hand"?

3. Consider the one-way ANOVA model (example b)) from class, version 2. Suppose, however, that there are 4 treatments (groups) and the sample sizes are respectively 2,1,1,3 for treatments 1 through 4. Show that the perpendicular projection matrix onto $C(\mathbf{X})$ (for observations in \mathbf{Y} listed in the order of group index) is

$$\begin{bmatrix} .5 & .5 & 0 & 0 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .3\bar{3} & .3\bar{3} & .3\bar{3} \\ 0 & 0 & 0 & 0 & .3\bar{3} & .3\bar{3} & .3\bar{3} \\ 0 & 0 & 0 & 0 & .3\bar{3} & .3\bar{3} & .3\bar{3} \end{bmatrix}$$

Do this two ways. First apply the following theorem (Theorem B.33 of Christensen).

Theorem B.33 \mathbf{M} is a perpendicular projection matrix onto $C(\mathbf{M})$ if and only if $\mathbf{M}\mathbf{M} = \mathbf{M}$ and $\mathbf{M}' = \mathbf{M}$.

(You'll need to reason that every column of \mathbf{X} can be written as a linear combination of columns of \mathbf{M} and vice versa, so that $C(\mathbf{X}) = C(\mathbf{M})$.)

Then secondly, use the construction for \mathbf{P}_X (involving a generalized inverse) given in class.

4. In the context of Problem 3, Suppose that $\mathbf{Y}' = (2, 1, 4, 6, 3, 5, 4)$.

- Find $\hat{\mathbf{Y}}$, the least squares estimate of $E\mathbf{Y} = \mathbf{X}\boldsymbol{\beta}$. Show that there is no sensible way to identify an "ordinary least squares estimate of $\boldsymbol{\beta}$ " by finding two different vectors \mathbf{b} with $\mathbf{X}\mathbf{b} = \hat{\mathbf{Y}}$.
- Use R and compute all of $\hat{\mathbf{Y}}$, $\mathbf{Y} - \hat{\mathbf{Y}}$, $\hat{\mathbf{Y}}'(\mathbf{Y} - \hat{\mathbf{Y}})$, $\mathbf{Y}'\mathbf{Y}$, $\hat{\mathbf{Y}}'\hat{\mathbf{Y}}$, and $(\mathbf{Y} - \hat{\mathbf{Y}})'(\mathbf{Y} - \hat{\mathbf{Y}})$.

5. Stat 542 and Stat 447 are supposed to cover multivariate distributions, and in particular multivariate normal distributions. Chapters 3 and 4 of Rencher cover that material and there are some summary facts in Appendix 7.1 of the 2004 Stat 511 Course Outline on this material. Review those if you need to do so. Then answer the following (making use of R to do matrix calculations). Suppose that $\mathbf{Y} \sim \text{MVN}_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu}' = (2, 2, 0)$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

- What is the marginal distribution of y_3 ?
- What is the (marginal joint) distribution of y_1 and y_3 ?
- What is the conditional distribution of y_3 given that $y_1 = 2$?
- What is the conditional distribution of y_3 given that $y_1 = 2$ and $y_2 = -1$?
- What is the conditional distribution of y_1 and y_3 , given that $y_2 = -1$?
- What are the correlations ρ_{12} , ρ_{13} , and ρ_{23} ?
- What is the joint distribution of $u = y_1 - y_2 + y_3$ and $v = 3y_1 + y_2 + 1$?

6. (Koehler) Use the `eigen()` function in R to compute the eigenvalues and eigenvectors of

$$\mathbf{V} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Then use R to find an "inverse square root" of this matrix. That is, find a symmetric matrix \mathbf{W} such $\mathbf{W}\mathbf{W} = \mathbf{V}^{-1}$. See slides 87-89 and 91 of Koehler's notes or Rencher for help with this. Koehler's Result 1.12 is Rencher's Theorem 2.12D.

7. (Koehler) Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$$

Obviously, these matrices are nearly identical. Use R and compute the determinants and inverses of these matrices. (Note that $\mathbf{A}^{-1} \approx -3\mathbf{B}^{-1}$ even though the original two matrices are nearly the same. This shows that small changes in the in the elements of nearly singular matrices can have big effects on some matrix operations.)

8. $\mathbf{c}'\boldsymbol{\beta}$ is estimable exactly when $\mathbf{c} \in C(\mathbf{X}')$. This occurs exactly when $\mathbf{c} = \mathbf{P}_{\mathbf{X}'}\mathbf{c}$, that is when \mathbf{c} is its own projection onto $C(\mathbf{X}')$. Clearly, $\mathbf{P}_{\mathbf{X}'} = \mathbf{X}'(\mathbf{X}\mathbf{X}')^{-1}\mathbf{X}$. Use R and find this matrix for the situation of Problem 3. Then use this matrix and R to decide which of the following linear combinations of parameters are estimable in this example:

$$\tau_1, \mu + \tau_1, \mu + \tau_1 + \tau_2, 2\mu + \tau_1 + \tau_2, \tau_1 - \tau_2, \text{ and } (\tau_1 - \tau_2) - (\tau_3 - \tau_4)$$

For those that are estimable, find the 7×1 row vector $\mathbf{c}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ that when multiplied by \mathbf{Y} produces the ordinary least squares estimate of $\mathbf{c}'\boldsymbol{\beta}$.

9. Twice now you've been asked to compute projection matrices in R. It seems like it would be helpful to “automate” this. Have a look at Chapter 10 of *An Introduction to R* (see [Help→Manuals \(in PDF\)](#) after starting R). Write a function (call it, say, `project`) that for an input matrix produces a matrix that projects vectors onto the column space of the input matrix. Test your function by running it on both \mathbf{X} and \mathbf{X}' for the situation of Problem 3.

10. Consider the (non-full-rank) “effects model” for the 2×2 factorial (with 2 observations per cell) called example d in lecture.

a) Determine which of the parametric functions below are estimable.

$$\alpha_1, \alpha_1 - \alpha_2, \mu + \alpha_1 + \beta_1, \mu + \alpha_1 + \beta_1 + \alpha\beta_{11}, \alpha\beta_{11}, \alpha\beta_{12} - \alpha\beta_{11} - (\alpha\beta_{22} - \alpha\beta_{21})$$

For those that are estimable, find the 8×1 row vector $\mathbf{c}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ that when multiplied by \mathbf{Y} produces the ordinary least squares estimate of $\mathbf{c}'\boldsymbol{\beta}$.

b) For the parameter vector $\boldsymbol{\beta}$ written in the order used in class, consider the hypothesis

$$H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{0}, \text{ for}$$

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

Is this hypothesis “testable”? Explain.

11. Consider the quadratic regression of part a) of Problem 1. Write the hypothesis that the pairs (x_{11}, x_{21}) and (x_{12}, x_{22}) have the same mean response in the form $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}$.