

Stat 511 HW#1 Spring 2004

1. Write out the following models of elementary/intermediate statistical analysis in the matrix form $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$:

a) A one-variable quadratic polynomial regression model

$$y_i = \alpha_0 + \alpha_1 x_i + \alpha_2 x_i^2 + \varepsilon_i$$

for $i = 1, 2, \dots, 5$

b) A two-factor ANOCVA model without interactions

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma(x_{ijk} - \bar{x}_{\dots}) + \varepsilon_{ijk}$$

for $i = 1, 2, j = 1, 2,$ and $k = 1, 2.$

2. Rencher, Problem 2.46. (See, for example, panels 162-168 of Koehler's notes or page 31 of Rencher.) Then load the MASS package in R and use the function `ginv()` provided by that package to find a generalized inverse. Does R return either of the generalized inverses you found "by hand"?

3. Consider the one-way ANOVA model (example b)) from class, version 2. Suppose, however, that there are 4 treatments (groups) and the sample sizes are respectively 2,1,1,2 for treatments 1 through 4. Show that the perpendicular projection matrix onto $C(\mathbf{X})$ (for observations in \mathbf{Y} listed in the order of group index) is

$$\begin{bmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & .5 \\ 0 & 0 & 0 & 0 & .5 & .5 \end{bmatrix}$$

Do this two ways. First apply the following theorem (Theorem B.33 of Christensen).

Theorem B.33 \mathbf{M} is a perpendicular projection matrix onto $C(\mathbf{M})$ if and only if $\mathbf{M}\mathbf{M} = \mathbf{M}$ and $\mathbf{M}' = \mathbf{M}$.

(You'll need to reason that every column of \mathbf{X} can be written as a linear combination of columns of \mathbf{M} and vice versa, so that $C(\mathbf{X}) = C(\mathbf{M})$.)

Then secondly, use the construction for $\mathbf{P}_{\mathbf{X}}$ (involving a generalized inverse) given in class.

4. In the context of Problem 3, Suppose that $\mathbf{Y}' = (2, 1, 4, 6, 3, 5)$.

a) Find $\hat{\mathbf{Y}}$, the least squares estimate of $E\mathbf{Y} = \mathbf{X}\boldsymbol{\beta}$. Show that there is no sensible way to identify an "ordinary least squares estimate of $\boldsymbol{\beta}$ " by finding two different vectors \mathbf{b} with $\mathbf{X}\mathbf{b} = \hat{\mathbf{Y}}$.

b) Use R and compute all of $\hat{\mathbf{Y}}, \mathbf{Y} - \hat{\mathbf{Y}}, \hat{\mathbf{Y}}'(\mathbf{Y} - \hat{\mathbf{Y}}), \mathbf{Y}'\mathbf{Y}, \hat{\mathbf{Y}}'\hat{\mathbf{Y}}$, and $(\mathbf{Y} - \hat{\mathbf{Y}})'(\mathbf{Y} - \hat{\mathbf{Y}})$.

5. Stat 542 and Stat 447 are supposed to cover multivariate distributions, and in particular multivariate normal distributions. Chapters 3 and 4 of Rencher cover that material and there are some summary facts in Appendix 7.1 of the Stat 511 Course Outline on this material. Review those if you need to do so. Then answer the following (making use of R to do matrix calculations).

Suppose that $\mathbf{Y} \sim \text{MVN}_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu}' = (0, 1, 0)$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 10 \end{pmatrix}$$

- What is the marginal distribution of y_3 ?
- What is the (marginal joint) distribution of y_1 and y_3 ?
- What is the conditional distribution of y_3 given that $y_1 = 2$?
- What is the conditional distribution of y_3 given that $y_1 = 2$ and $y_2 = -1$?
- What is the conditional distribution of y_1 and y_3 , given that $y_2 = -1$?
- What are the correlations ρ_{12}, ρ_{13} , and ρ_{23} ?
- What is the joint distribution of $u = y_1 - y_2 + y_3$ and $v = 3y_1 + y_2 + 1$?

6. (Koehler) Use the `eigen()` function in R to compute the eigenvalues and eigenvectors of

$$\mathbf{V} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Then use R to find an “inverse square root” of this matrix. That is, find a symmetric matrix \mathbf{W} such $\mathbf{W}\mathbf{W} = \mathbf{V}^{-1}$. See slides 87-89 and 91 of Koehler's notes or page 48 of Rencher for help with this. Koehler's Result 1.12 is Rencher's Theorem 2.12D, page 46.

7. (Koehler) Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$$

Obviously, these matrices are nearly identical. Use R and compute the determinants and inverses of these matrices. (Note that $\mathbf{A}^{-1} \approx -3\mathbf{B}^{-1}$ even though the original two matrices are nearly the same. This shows that small changes in the in the elements of nearly singular matrices can have big effects on some matrix operations.)