1. Use (mixed model) thermocouple calibration problem of HW8.

(a) Results match those obtained in 3(e) of HW8. The ML estimates are: $\widehat{\sigma}^2_\gamma = 0.0629$, $\widehat{\sigma}^2_\delta = 0.4441$ and $\widehat{\sigma}^2 = 0.3733$.

```
> minuslLstar(c(1,1,1),Y)
[1,] 15.55036
> minuslLstar(c(1,1,.25),Y)
[1,] 14.30018
> optim(c(.07,.45,.37),minuslLstar,Y=Y,hessian=TRUE)$par
[1] 0.06291014 0.44413348 0.37327909
```

(b) The REML estimates are: $\widehat{\sigma}^2_\gamma = 0.3438$, $\widehat{\sigma}^2_\delta = 0.8087$, and $\widehat{\sigma}^2 = 0.2848$. REML estimates for σ^2_γ and σ^2_δ are bigger than ML estimates.

```
Px <- X%*%ginv(t(X)%*%X)%*%t(X)
N <- diag(rep(1,9))-Px
B <- N[1:7,]
> qr(B)$rank
[1] 7
> optim(c(.07,.45,.37),minuslLstar2,Y=Y,B=B,hessian=TRUE)$par
[1] 0.3437852 0.8087115 0.2848388
```

(c) $C\widehat{\beta}_{MLE} = C\widehat{\beta}_{REMLE} = (\widehat{\alpha} = 99.4551, \widehat{\beta} = 9.7936)$

```
s2 <- optim(c(.07,.45,.37),minuslLstar,Y=Y,hessian=TRUE)$par       # ML estimates
s2 <- optim(c(.07,.45,.37),minuslLstar2,Y=Y,B=B,hessian=TRUE)$par  # REML estimates
Vhat <- kronecker(I3,((s2[1]*J3)+(s2[2]*M)+(s2[3]*I3)))
C <- diag(c(1,1)); b <- ginv(t(X)%*%solve(Vhat)%*%X)%*%t(X)%*%solve(Vhat)%*%Y
Cb <- C%*%b
```

(d)
```
Var <- C%*%ginv(t(X)%*%solve(Vhat)%*%X)%*%t(C); stderr <- sqrt(diag(Var))
> stderr                        # for ML estimates
[1] 0.3198840 0.4029905
> stderr                        # for REML estimates
[1] 0.4203276 0.5296469
```

(e)
```
G <- diag(c(rep(s2[1],3),rep(s2[2],3)))
Z <- matrix(c(rep(1,3),rep(0,9),rep(1,3),rep(0,9),rep(1,3),0,1,4,rep(0,9),0,1,4,rep(0,9),0,1,4),9,6)
B <- X%*%ginv(t(X)%*%solve(Vhat)%*%X)%*%t(X)%*%solve(Vhat)
P <- solve(Vhat)%*%(diag(rep(1,9))-B)
uhathat <- G%*%t(Z)%*%P%*%Y
```

$\hat{\hat{u}}_{MLE}$	-0.02172176	0.1511162	-0.1293945	-0.6478563	-0.2366669	0.8845232
$\hat{\hat{u}}_{REMLE}$	-0.02999012	0.4881394	-0.4581493	-0.6637030	-0.3404386	1.0041420

(f)
```
Var <- G - G%*%t(Z)%*%P%*%Z%*%G; stderr <- sqrt(diag(Var))
> stderr                                                             # for ML estimates
[1] 0.2323972 0.2323972 0.2323972 0.4056949 0.4056949 0.4056949
> stderr                                                             # for REML estimates
[1] 0.4405919 0.4405919 0.4405919 0.5358329 0.5358329 0.5358329
```

(g)
```
cvector <- matrix(c(1,0),2,1); svector <- matrix(c(1,0,0,0,0,0),6,1)   # alpha1
cvector <- matrix(c(0,1),2,1); svector <- matrix(c(0,0,0,1,0,0),6,1)   # beta1
l <- t(cvector)%*%b+t(svector)%*%uhathat
at <- t(cvector)%*%t(X)%*%ginv(X%*%t(X))
Var <- t(cvector)%*%ginv(t(X)%*%solve(Vhat)%*%X)%*%cvector+
       t(svector)%*%G%*%t(Z)%*%P%*%Z%*%G%*%svector - 2*at%*%B%*%Z%*%G%*%svector
stderr <- sqrt(diag(Var))
```

	MLE	std error	REMLE	std error
α_1	99.4334	0.2632	99.4251	0.3117
β_1	9.1457	0.3819	9.1299	0.5128

2.
```
thermo <- read.table("hw09.txt",header=T)
gd <- groupedData(y~temp|group,data=thermo)
fm1 <- lme(y~temp,data=gd,random=~1+temp|group,method="ML")
fm2 <- lme(y~temp,data=gd,random=~1+temp|group,method="REML")
fm3 <- update(fm1,random=pdDiag(~1+temp),method="ML")
fm4 <- update(fm2,random=pdDiag(~1+temp),method="REML")

> fixed.effects(fm3)          # ML results in 1(c)
(Intercept)        temp
   99.45513     9.79359
> random.effects(fm3)         # ML results in 1(e)
  (Intercept)       temp
1 -0.02181365 -0.6478858
2  0.15202592 -0.2369421
3 -0.13021227  0.8848279
> coefficients(fm3)
  (Intercept)      temp
1    99.43331  9.145704       # This row was obtained in 1(g) using ML
2    99.60715  9.556648
3    99.32492 10.678418
```

In the function `intervals()` the point estimates for the variance components correspond to the square root of the estimates obtained in 1(a). (e.g., `sd((Intercept))` $= \sqrt{\widehat{\sigma}^2_\gamma} = \sqrt{0.0629} \approx 0.2516$). Note that given the size of the data, the confidence intervals for the ML estimates of the variance components are very wide.

```
> intervals(fm3)
Approximate 95% confidence intervals
 Fixed effects:
                lower     est.     upper
(Intercept) 98.632658 99.45513 100.27760
temp         8.757383  9.79359  10.82980

 Random Effects:
  Level: group
                       lower      est.       upper
sd((Intercept)) 4.895553e-05 0.2516244 1293.313462
sd(temp)        2.272768e-01 0.6666714    1.955548

 Within-group standard error:
    lower      est.     upper
0.1449576 0.6106299 2.5722616
```

Results for REML estimates are obtained using the same functions with the model stored in `fm4`.