1. A 90% confidence interval for σ based on carrying over the linear model result that $SSE/\sigma^2 \sim \chi^2_{n-k}$ is $(11.1, 40.5)$. A 90% confidence interval for σ based on "profile likelihood material" is $(9.3, 24.5)$ and it is obtained considering

$$\left(\sigma^2| -\frac{n}{2}log(\sigma^2) - \frac{SSE}{2\sigma^2} > -\frac{n}{2}log(\frac{SSE}{n}) - \frac{n}{2} - \frac{1}{2}\chi^2_1\right) = \left(-3\,log(\sigma^2) - \frac{584.0044}{\sigma^2} > -20.16666\right)$$

```
ll <- sqrt(ss(b1,b2)/qchisq(0.95,4))     # SSE/sigma^2 ~ chi-squared
ul <- sqrt(ss(b1,b2)/qchisq(0.05,4))
sigma2 <- seq(50,2000,0.01)              # profile likelihood
L <- -3*log(sigma2)-584.0044/sigma2
tmp <- cbind(sigma2,L)
tmp <- matrix(tmp[L>-20.16666],ncol=2)
sqrt(min(tmp[,1])) ; sqrt(max(tmp[,1]))
```

2. This 90% confidence interval for β_1 is similar to the one visually obtained in part (g) of HW7.

```
> confint.nls(BOD.fm, level=0.90)
            5%         95%
b1 188.0555907 246.2066097
b2   0.3491401   0.8997058
```

3. $Y = X\beta + Zu + \epsilon$

(a)
$$\begin{bmatrix} y_{11}\\ y_{12}\\ y_{13}\\ y_{21}\\ y_{22}\\ y_{23}\\ y_{31}\\ y_{32}\\ y_{33}\end{bmatrix} = \begin{bmatrix} 1&0\\1&1\\1&4\\1&0\\1&1\\1&4\\1&0\\1&1\\1&4\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} + \begin{bmatrix} 1&0&0&0&0&0\\ 1&0&0&1&0&0\\ 1&0&0&4&0&0\\ 0&1&0&0&0&0\\ 0&1&0&0&1&0\\ 0&1&0&0&4&0\\ 0&0&1&0&0&0\\ 0&0&1&0&0&1\\ 0&0&1&0&0&4\end{bmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\\ \delta_1\\ \delta_2\\ \delta_3\end{bmatrix} + \begin{bmatrix}\epsilon_{11}\\ \epsilon_{12}\\ \epsilon_{13}\\ \epsilon_{21}\\ \epsilon_{22}\\ \epsilon_{23}\\ \epsilon_{31}\\ \epsilon_{32}\\ \epsilon_{33}\end{bmatrix}$$

(b)
$$E(Y) = X\beta = \begin{bmatrix}\alpha\\ \alpha+\beta\\ \alpha+4\beta\\ \alpha\\ \alpha+\beta\\ \alpha+4\beta\\ \alpha\\ \alpha+\beta\\ \alpha+4\beta\end{bmatrix} \text{ and } Var(Y) = \begin{bmatrix} A&0&0\\0&A&0\\0&0&A\end{bmatrix} \text{ where } A = \begin{bmatrix}\sigma^2_\gamma+\sigma^2 & \sigma^2_\gamma & \sigma^2_\gamma\\ \sigma^2_\gamma & \sigma^2_\gamma+\sigma^2_\delta+\sigma^2 & \sigma^2_\gamma+4\sigma^2_\delta\\ \sigma^2_\gamma & \sigma^2_\gamma+4\sigma^2_\delta & \sigma^2_\gamma+16\sigma^2_\delta+\sigma^2\end{bmatrix}$$

(c) $\widehat{\alpha} = 99.45513$ and $\widehat{\beta} = 9.79359$ in both cases. Note that in both cases the variance components have the same proportional relation.

```
X <- matrix(c(rep(1,9),rep(c(0,1,4),3)),9,2)
Y <- matrix(c(99.8,108.1,136.0,100.3,109.5,137.7,98.3,110.1,142.2),9,1)
#sg <- 1; sd <- 1; s <- 0.25     # case 1
sg <- 4; sd <- 4; s <- 1        # case 2 (times sigma^2)
A <- matrix(c(sg+s,sg,sg,sg,sg+sd+s,sg+4*sd,sg,sg+4*sd,sg+16*sd+s),3,3)
V <- matrix(0,9,9); V[1:3,1:3] <- A; V[4:6,4:6] <- A; V[7:9,7:9] <- A
b <- solve(t(X)%*%solve(V)%*%X)%*%t(X)%*%solve(V)%*%Y
```

(d) The BLUE for (α, β) are the same as those obtained in (c). (Use `sg <- 1; sd <- 1; s <- 1` and follow steps in (c)).

(e) For each V compute Y^* and $L^*(\sigma^2)$.

$\sigma^2_\gamma = 1, \sigma^2_\delta = 1, \sigma^2 = 0.25$ $\quad logL^*(\sigma^2) = -14.30018$ $\quad$ larger value of profile loglikelihood
$\sigma^2_\gamma = 1, \sigma^2_\delta = 1, \sigma^2 = 1$ $\quad logL^*(\sigma^2) = -15.55036$

```
Ystar <- X%*%ginv(t(X)%*%solve(V)%*%X)%*%t(X)%*%solve(V)%*%Y
logLstar <- (-9/2)*log(2*pi)-.5*log(det(V))-.5*(t(Y-Ystar)%*%solve(V)%*%(Y-Ystar))
```