1. ("$\Leftarrow$") Assume $\underline{c} \in C(X')$. For $\underline{c} \in C(X')$ and $\underline{d} \in C(X')^{\perp}$ then $\underline{d} \perp \underline{c}$ by definiton of $C(X')^{\perp}$.

   ("$\Rightarrow$") Assume "$\underline{c}$ is special". Write $\underline{c} = P_{X'}\underline{c} + (I - P_{X'})\underline{c} = P_{X'}\underline{c} + \underline{d}^*$.

   If $\underline{d}^* = (I - P_{X'})\underline{c} = \underline{0}$, then $P_{X'}\underline{c} = \underline{c}$ and $\underline{c} \in C(X')$.

   Suppose that $\underline{d}^* = (I - P_{X'})\underline{c} \neq \underline{0}$. Then $\underline{d}^* \in C(X')^{\perp}$.

   Compute $\underline{c}'\underline{d}^* = (P_{X'}\underline{c} + \underline{d}^*)'\underline{d}^* = \underline{c}'P_{X'}\underline{d}^* + \underline{d}^{*'}\underline{d}^* = \underline{c}'\underline{0} + \underline{d}^{*'}\underline{d}^* = \underline{d}^{*'}\underline{d}^* > 0$ because $\underline{d}^* \neq \underline{0}$.

   But this contradicts "$\underline{c}$ is special". Then $\underline{d}^* = 0$ and $\underline{c} = P_{X'}\underline{c}$. Therefore, $\underline{c} \in C(X')$.

2. For V_1, $(\hat{Y}^*)' = (5/3, 5/3, 5/3, 17, 11, 11)'$ and $C\widehat{\underline{\beta}} = (5/3, 17, 11)'$.

   $$Cov(\hat{Y}^*) = \sigma^2 X(W'W)^- X' = \sigma^2 \begin{bmatrix} 4/9 & 4/9 & 4/9 & 0 & 0 & 0 \\ 4/9 & 4/9 & 4/9 & 0 & 0 & 0 \\ 4/9 & 4/9 & 4/9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{bmatrix}. \; Cov(C\widehat{\underline{\beta}}) = \sigma^2 C(W'W)^- C' = \sigma^2 \begin{bmatrix} 4/9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   $$Cov(\hat{Y}) = \sigma^2 P_X V_1 P_X = \sigma^2 \begin{bmatrix} 2/3 & 2/3 & 2/3 & 0 & 0 & 0 \\ 2/3 & 2/3 & 2/3 & 0 & 0 & 0 \\ 2/3 & 2/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{bmatrix}. \; Cov(C\widehat{\underline{\beta}}_{OLS}) = \sigma^2 C(X'X)^- X' V_1 X (X'X)^- C' = \sigma^2 \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   In Aitken model, generalized least squares estimators have smallest variance among linear unbiased estimators.

   For V_2, $(\hat{Y}^*)' = (2, 2, 2, 17, 11, 11)'$ and $C\widehat{\underline{\beta}} = (2, 17, 11)'$.

3. For Aitken BLUE: $Cov(C\widehat{\underline{\beta}}) = C(W'W)^- C' = \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{bmatrix}$.

   For Gauss-Markov BLUE: $Cov(C\widehat{\underline{\beta}}_{OLS}) = C(X'X)^- C' = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$.

   In the Gauss-Markov model all observations are uncorrelated. In this Aitken model observations are uncorrelated between groups, but they are correlated within groups, reducing the amount of "new information" provided by each additional entry of the Y vector. Therefore, $Cov(C\widehat{\underline{\beta}})$ for the Gauss Markov model is better than for the proposed Aitken model.

4. $\mu + \alpha_1 + \beta_1 + \alpha\beta_{11}$ and $(\alpha\beta_{12} - \alpha\beta_{11}) - (\alpha\beta_{22} - \alpha\beta_{21})$ are estimable.

   Let $\underline{a}' = (1, 0, 0, 0, 0, 0, 0, 0)'$. Then $E(\underline{a}'Y) = E(Y_{111}) = \mu + \alpha_1 + \beta_1 + \alpha\beta_{11}$.

   Let $\underline{a}' = (-1, 0, 1, 0, 1, 0, -1, 0)'$. Then $E(\underline{a}'Y) = E(-Y_{111} + Y_{121} + Y_{211} - Y_{221}) = (\alpha\beta_{12} - \alpha\beta_{11}) - (\alpha\beta_{22} - \alpha\beta_{21})$.

   * Note that these vectors $\underline{a}$ are not unique.