STAT 511 **Solutions to Homework 2** **Spring 2003**

1. The function `solve()` fails to find the inverse of nearly singular matrices. Below there is an example of a function called `ginverse` that uses the singular value decomposition to obtain a generalized inverse.

```
ginverse <- function(X, tol = sqrt(.Machine$double.eps)) {
  s <- svd(X)
  nz <- s$d > tol * s$d[1]
  if(any(nz)) s$v[,nz]%*%(t(s$u[,nz])/s$d[nz]) else X*0
}

> A <- matrix(c(4,4.001,4.001,4.002),2,2)
> B <- matrix(c(4,4.001,4.001,4.002001),2,2)
> ginverse(A)
         [,1]     [,2]
[1,] -4002000  4001000
[2,]  4001000 -4000000
> ginverse(B)
         [,1]     [,2]
[1,]  1334000 -1333667
[2,] -1333667  1333333
> 3*ginverse(B)
         [,1]     [,2]
[1,]  4002001 -4001000
[2,] -4001000  4000000
> det(A);det(B)         # determinants of A and B, respectively.
[1] -1e-06
[1] 3e-06
```

2. Using the function `eigen()` to obtain eigenvalues and eigenvectors of a matrix V we are able to compute its inverse square root, $W = V^{-1/2} = UD^{-1/2}U'$. Note that $WW = V^{-1}$.

```
> V <- matrix(c(3,-1,1,-1,5,-1,1,-1,3),3,3,byrow=T)
> EV <- eigen(V)
> W <- EV$vectors%*%diag(1/sqrt(EV$values))%*%t(EV$vectors)
> W
            [,1]       [,2]        [,3]
[1,]  0.61404486 0.05636733 -0.09306192
[2,]  0.05636733 0.46461562  0.05636733
[3,] -0.09306192 0.05636733  0.61404486
> W%*%W
            [,1]       [,2]        [,3]
[1,]  0.38888889 0.05555556 -0.11111111
[2,]  0.05555556 0.22222222  0.05555556
[3,] -0.11111111 0.05555556  0.38888889
> solve(V)
            [,1]       [,2]        [,3]
[1,]  0.38888889 0.05555556 -0.11111111
[2,]  0.05555556 0.22222222  0.05555556
[3,] -0.11111111 0.05555556  0.38888889
```

3. In this question we use the function produced in question 5 to compute P_X.

```
> X <- matrix(nrow=6,ncol=4,c(rep(c(1,1,0,0),3),1,0,1,0,rep(c(1,0,0,1),2)),byrow=T)
> Y <- c(2,1,3,17,10,12)
> Px <- project(X)
> Yhat <- Px%*%Y
```

(a) $\hat{Y}' = (2, 2, 2, 17, 11, 11)'$.

(b) $(Y - \hat{Y})' = (0, -1, 1, 0, -1, 1)'$.

(c) $\hat{Y}'(Y - \hat{Y}) = 0$.

(d) $Y'Y = 547$.

(e) $\hat{Y}'\hat{Y} = 543$.

(f) $(Y - \hat{Y})'(Y - \hat{Y}) = 4$.

4. $\mu + \alpha_1$, $2\mu + \alpha_1 + \alpha_2$ and $\alpha_1 - \alpha_2$ are estimable.

   (a) Let $\underline{c}' = (1, 0, 0, 0)'$. $P_{X'}\underline{c} \neq \underline{c}$, then μ is not estimable.

   (b) Let $\underline{c}' = (0, 1, 0, 0)'$. $P_{X'}\underline{c} \neq \underline{c}$, then α_1 is not estimable.

   (c) Let $\underline{c}' = (1, -1, 0, 0)'$. $P_{X'}\underline{c} \neq \underline{c}$, then $\mu - \alpha_1$ is not estimable.

   (d) Let $\underline{c}' = (1, 1, 0, 0)'$. $P_{X'}\underline{c} = \underline{c}$, then $\mu + \alpha_1$ is estimable.
   - $\underline{\rho}' = \underline{c}'(X'X)^{-}X' = (1/3, 1/3, 1/3, 0, 0, 0)'$.
   - $\underline{\rho}'Y = \widehat{\underline{c}'\underline{\beta}}_{OLS} = 2$.

   (e) Let $\underline{c}' = (1, 1, 1, 0)'$. $P_{X'}\underline{c} \neq \underline{c}$, then $\mu + \alpha_1 + \alpha_2$ is not estimable.

   (f) Let $\underline{c}' = (2, 1, 1, 0)'$. $P_{X'}\underline{c} = \underline{c}$, then $2\mu + \alpha_1 + \alpha_2$ is estimable.
   - $\underline{\rho}' = \underline{c}'(X'X)^{-}X' = (1/3, 1/3, 1/3, 1, 0, 0)'$.
   - $\underline{\rho}'Y = \widehat{\underline{c}'\underline{\beta}}_{OLS} = 19$.

   (g) Let $\underline{c}' = (0, 1, -1, 0)'$. $P_{X'}\underline{c} = \underline{c}$, then $\alpha_1 - \alpha_2$ is estimable.
   - $\underline{\rho}' = \underline{c}'(X'X)^{-}X' = (1/3, 1/3, 1/3, -1, 0, 0)'$.
   - $\underline{\rho}'Y = \widehat{\underline{c}'\underline{\beta}}_{OLS} = -15$.

   The results are sensible since the vectors $\underline{\rho}'$ when multiplied by Y produce linear combinations of the sample means of the different groups.

5. `project <- function(A) {A%*%ginverse(t(A)%*%A)%*%t(A)}`

```
> Pxt <- project(t(X))
> Pxt
     [,1]  [,2]  [,3]  [,4]
[1,] 0.75  0.25  0.25  0.25
[2,] 0.25  0.75 -0.25 -0.25
[3,] 0.25 -0.25  0.75 -0.25
[4,] 0.25 -0.25 -0.25  0.75
> round(Pxt%*%t(X),1)
     [,1] [,2] [,3] [,4] [,5] [,6]
[1,]    1    1    1    1    1    1
[2,]    1    1    1    0    0    0
[3,]    0    0    0    1    0    0
[4,]    0    0    0    0    1    1
```