STAT 511 **Solutions to Homework 1** **Spring 2003**

1. Exercise 1.1. Write the following models in matrix notation:

   (c) Two-way analysis of covariance (ACOVA) with no interaction.

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma x_{ijk} + e_{ijk}, \quad i = 1,2,3, \;\; j = 1,2, \;\; k = 1,2.$$

$$\begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \\ y_{311} \\ y_{312} \\ y_{321} \\ y_{322} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & x_{111} \\ 1 & 1 & 0 & 0 & 1 & 0 & x_{112} \\ 1 & 1 & 0 & 0 & 0 & 1 & x_{121} \\ 1 & 1 & 0 & 0 & 0 & 1 & x_{122} \\ 1 & 0 & 1 & 0 & 1 & 0 & x_{211} \\ 1 & 0 & 1 & 0 & 1 & 0 & x_{212} \\ 1 & 0 & 1 & 0 & 0 & 1 & x_{221} \\ 1 & 0 & 1 & 0 & 0 & 1 & x_{222} \\ 1 & 0 & 0 & 1 & 1 & 0 & x_{311} \\ 1 & 0 & 0 & 1 & 1 & 0 & x_{312} \\ 1 & 0 & 0 & 1 & 0 & 1 & x_{321} \\ 1 & 0 & 0 & 1 & 0 & 1 & x_{322} \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \gamma \end{bmatrix} + \begin{bmatrix} e_{111} \\ e_{112} \\ e_{121} \\ e_{122} \\ e_{211} \\ e_{212} \\ e_{221} \\ e_{222} \\ e_{311} \\ e_{312} \\ e_{321} \\ e_{322} \end{bmatrix}$$

   (d) Multiple polynomial regression.

$$y_i = \beta_{00} + \beta_{10}x_{i1} + \beta_{01}x_{i2} + \beta_{20}x_{i1}^2 + \beta_{02}x_{i2}^2 + \beta_{11}x_{i1}x_{i2} + e_i, \quad i = 1, \ldots, 6.$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{12} & x_{11}^2 & x_{12}^2 & x_{11}x_{12} \\ 1 & x_{21} & x_{22} & x_{21}^2 & x_{22}^2 & x_{21}x_{22} \\ 1 & x_{31} & x_{32} & x_{31}^2 & x_{32}^2 & x_{31}x_{32} \\ 1 & x_{41} & x_{42} & x_{41}^2 & x_{42}^2 & x_{41}x_{42} \\ 1 & x_{51} & x_{52} & x_{51}^2 & x_{52}^2 & x_{51}x_{52} \\ 1 & x_{61} & x_{62} & x_{61}^2 & x_{62}^2 & x_{61}x_{62} \end{bmatrix} \begin{bmatrix} \beta_{00} \\ \beta_{10} \\ \beta_{01} \\ \beta_{20} \\ \beta_{02} \\ \beta_{11} \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix}$$

2. Exercise 1.11. For a linear model $Y = X\beta + e$, $E(e) = 0$, $Cov(e) = \sigma^2 I$, the residuals are $\hat{e} = (I - M)Y$, where M is the perpendicular projection operator onto $C(X)$.

   (a) $E(\hat{e}) = (I - M)E(Y) = (I - M)X\beta = (X - MX)\beta = (X - X)\beta = 0.$

   (b) $Cov(\hat{e}) = (I - M)Cov(Y)(I - M)' = (I - M)\sigma^2 I(I - M)' = \sigma^2(I - M)(I - M)' = \sigma^2(I - M).$

   (c) $Cov(\hat{e}, MY) = Cov((I - M)Y, MY) = (I - M)\sigma^2 IM' = \sigma^2(I - M)M = 0.$

   (d)
$$\begin{aligned} E(\hat{e}'\hat{e}) &= E(Y'(I - M)'(I - M)Y) \\ &= E(Y'(I - M)Y) \\ &= tr((I - M)\sigma^2 I) + \beta'X'(I - M)X\beta \quad \textit{from Thm. } 1.3.2. \\ &= \sigma^2 tr(I - M) + \beta'X'(X - MX)\beta \\ &= \sigma^2 rank(I - M) + \beta'X'(X - X)\beta \\ &= \sigma^2[n - rank(X)] \end{aligned}$$

   (e) $\hat{e}'\hat{e} = Y'(I - M)'(I - M)Y = Y'(I - M)Y = Y'Y - Y'MY.$

   (f) Rewrite (c) and (e) in terms of $\hat{\beta}$.

      i. $Cov(\hat{e}, X\hat{\beta}) = 0.$

      ii. $\hat{e}'\hat{e} = Y'Y - (MY)'Y = Y'Y - \hat{\beta}'X'Y = (Y - X\hat{\beta})'Y.$

3. Exercise 1.5.2. Let $Y = (y_1, y_2, y_3)'$ with $Y \; N(\mu, V)$ where $\mu = (5, 6, 7)'$ and $V = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}$.

   (a) $y_1 \sim N(5, 2).$

   (b) $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \right).$

(c) $$y_3|y_1=u_1, y_2=u_2 \sim N\left(7+\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}^{-1}\begin{bmatrix}u_1-5\\ u_2-6\end{bmatrix}, 4-\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}^{-1}\begin{bmatrix}1\\ 2\end{bmatrix}\right) = N\left(u_1/2+2u_2/3+1/2, 13/6\right).$$

(d) $[y_3|y_1=u_1] \sim N(7+1(1/2)(u_1-5), 4-1(1/2)1) = N((1/2)(u_1+9), 7/2)$.

(e) $$\begin{bmatrix}y_1\\ y_2\end{bmatrix}\Bigg| y_3=u_3 \sim N\left(\begin{bmatrix}5\\ 6\end{bmatrix}+\begin{bmatrix}1\\ 2\end{bmatrix}[1/4][u_3-7], \begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}-\begin{bmatrix}1\\ 2\end{bmatrix}[1/4]\begin{bmatrix}1 & 2\end{bmatrix}\right) = N\left(\begin{bmatrix}u_3/4+13/4\\ u_3/2+5/2\end{bmatrix}, \begin{bmatrix}7/4 & -1/2\\ -1/2 & 2\end{bmatrix}\right)$$

(f) Find the correlations ρ_{12}, ρ_{13}, ρ_{23}.

i. $\rho_{12}=0/\sqrt{2(3)}=0$.

ii. $\rho_{13}=1/\sqrt{2(4)}=1/(2\sqrt{2})$.

iii. $\rho_{23}=2/\sqrt{3(4)}=1/\sqrt{3}$.

(g) $$Z \sim N\left(\begin{bmatrix}2 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}E(Y), \begin{bmatrix}2 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}Cov(Y)\begin{bmatrix}2 & 1\\ 1 & 1\\ 1 & 0\end{bmatrix}\right) = N\left(\begin{bmatrix}1\\ 0\end{bmatrix}, \begin{bmatrix}11 & 11\\ 11 & 15\end{bmatrix}\right).$$

4. $AG_1A=A$ and $AG_2A=A$ then G_1 and G_2 are generalized inverses for A.

```
> A <- matrix(nrow=3,ncol=4,c(4,1,2,0,1,1,5,15,3,1,3,5),byrow=T)
> G1 <- matrix(nrow=4,ncol=3,c(0,0,0,0,-1.5,2.5,0,.5,-.5,0,0,0),byrow=T)
> G2 <- matrix(nrow=4,ncol=3,c(.25,rep(0,8),-.15,0,.2),byrow=T)
> A%*%G1%*%A
     [,1] [,2] [,3] [,4]
[1,]    4    1    2    0
[2,]    1    1    5   15
[3,]    3    1    3    5
> A%*%G2%*%A
     [,1] [,2] [,3] [,4]
[1,]    4    1    2    0
[2,]    1    1    5   15
[3,]    3    1    3    5
```

5. Exercise 1.5.8. (b) Show that for M is the projection operator onto $C(X)$, where

$$X=\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 1\end{bmatrix}, \quad M=\begin{bmatrix}1/3 & 1/3 & 1/3 & 0 & 0 & 0\\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0\\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1/2 & 1/2\\ 0 & 0 & 0 & 0 & 1/2 & 1/2\end{bmatrix}$$

i. $C(X)$ is the set of all linear combinations of the columns of X.

$$a_1\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}+a_2\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}+a_3\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}+a_4\begin{bmatrix}0\\0\\0\\0\\1\\1\end{bmatrix}=\begin{bmatrix}a_1+a_2\\a_1+a_2\\a_1+a_2\\a_1+a_3\\a_1+a_4\\a_1+a_4\end{bmatrix}=\begin{bmatrix}a\\a\\a\\b\\c\\c\end{bmatrix}$$

Similarly, $C(M)$ also consists of vectors of the form $(a,a,a,b,c,c)'$. By inspection it is clear that M is symmetric ($M'=M$), and using R, we can verify that M is idempotent ($MM=M$). Therefore, by Proposition B.33, M is a perpendicular projection operator on $C(M)$, and since $C(M)=C(X)$, M is a perpendicular projection operator on $C(X)$.

Or, if we note that the orthogonal complement of $C(X)$ consists of vectors w of the form $(d,d,-2d,0,e,-e)'$ and that

- for $v=(a,a,a,b,c,c)'$ then $Mv=v$, and
- for $w=(d,d,-2d,0,e,-e)$ then $Mw=0$.

By Definition B.31, M is a perpendicular projection operator matrix onto $C(X)$ and by Proposition B.32 $C(X) = C(M)$.

ii. Using the singular value decomposition to obtain $(X'X)^-$ we compute $M = X(X'X)^-X'$, which is the projection operator onto $C(X)$ and, it matches the matrix proposed in the statement of this question.

```
> X <- matrix(nrow=6,ncol=4,c(rep(c(1,1,0,0),3),1,0,1,0,rep(c(1,0,0,1),2)),byrow=T)
> s <-svd(t(X)%*%X)
> d <- rep(0,length(s$d))
> for (i in 1:length(s$d)) {if (s$d[i] != 0) d[i] <- 1/s$d[i]}
> ginvXtX <- s$v%*%diag(d)%*%t(s$u)
> M <- X%*%ginvXtX%*%t(X)
> round(M,2)
     [,1] [,2] [,3] [,4] [,5] [,6]
[1,] 0.33 0.33 0.33    0  0.0  0.0
[2,] 0.33 0.33 0.33    0  0.0  0.0
[3,] 0.33 0.33 0.33    0  0.0  0.0
[4,] 0.00 0.00 0.00    1  0.0  0.0
[5,] 0.00 0.00 0.00    0  0.5  0.5
[6,] 0.00 0.00 0.00    0  0.5  0.5
```

6. For the same X as in question 5, suppose that $Y' = (2, 1, 3, 17, 10, 12)'$. Find $\hat{Y}$, the ordinary least squares estimate of $EY = X\underline{\beta}$. Show that there is no sensible way to identify the OLS estimate of $\underline{\beta}$, by finding two different vectors $\underline{b}$ with $X\underline{b} = \hat{Y}$.

Let G_1 and G_2 be two different generalized inverses for $(X'X)$, and let $\underline{b_i} = G_i X'Y$ for $i = 1, 2$.

$$G_1 = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 0 \\ -1/2 & 5/6 & 1/2 & 0 \\ -1/2 & 1/2 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}.$$

Note that $\hat{Y}' = X\underline{b_i} = (2, 2, 2, 7, 11, 11)'$ for $i = 1, 2$; but $\underline{b_1} = (11, -9, 6, 0)'$ and $\underline{b_2} = (0, 2, 17, 11)'$.