STAT 511 Solution to HW7 Spring 2008

1. (a) Using lmer, we get the estimates of the variance components. $\hat{\sigma}_\alpha^2 = 1.38694, \hat{\sigma}_\beta^2 = 0.24642$ and $\hat{\sigma}^2 = 1.11408$. The MCMC method gives us intervals for these estimates as follows:

```
                      lower    upper
log(sigma^2)     -0.2713032 1.629201
log(A:B.(In)) -130.8553019 2.852704
log(A.(In))   -337.9054621 6.853289
```

Based on this, the confidence interval for σ is (0.873, 2.258). The pooled variance estimate is 1.090. An exact confidence interval for σ^2 based on this pooled variance is $(SSE/\chi^2_{8,0.975}, SSE/\chi^2_{8,0.025})$. Then the confidence interval for σ is (0.71, 2.00), which is almost the same length with confidence limits with that given by MCMC.

(b) We fit a fixed effects model to this data and get the following result

```
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   8.0694     0.3293  24.503 8.22e-09 ***
A1           -0.6694     0.3293  -2.033   0.0765 .
A1:B1        -1.3500     0.9040  -1.493   0.1737
A2:B1         0.9111     0.5857   1.556   0.1584
A1:B2             NA         NA      NA       NA
A2:B2        -0.2389     0.5857  -0.408   0.6941
```

We get "Residual standard error: 1.044 on 8 degrees of freedom". Note that $1.044^2 \approx 1.090$.

(c) Use predict() for (b), the predict for the 13 data is:

```
6.050000 6.050000 7.400000 7.400000 7.400000 7.400000
9.650000 9.650000 8.500000 8.500000 8.066667 8.066667 8.066667
```

All these values are the sample mean in each level of B within A. For example, $6.05 = (6.0 + 6.1)/2$. Using fitted() in part (a), we get the following estimates:

```
6.735651 6.735651 7.208440 7.208440 7.208440 7.208440
8.884221 8.884221 8.531522 8.531522 8.354484 8.354484 8.354484
```

These estimates are different form what given by fixed effects model above.

2. (a) Based on the model, the EMS for these three factors are EMS(Machine) = $\sigma^2 + 25\sigma_\alpha^2 + 5\sigma_\beta^2$, EMS(Rolls) = $\sigma^2 + 5\sigma_\beta^2$, and EMS(Piecies) = σ^2. An estimate of $\hat{\sigma}_\alpha^2$ = (MS(Machine) − MS(Rolls))/25 = 37.17333 and an estimate of $\hat{\sigma}_\beta^2$ = (MS(Rolls) − MS(Piecies))/5 = 9.8. Then the Cochran-Satterthwaite confidence limits for σ_α^2 is

$$\left(\frac{\hat{\nu}\hat{\sigma}_\alpha^2}{\chi^2_{\hat{\nu},0.975}}, \frac{\hat{\nu}\hat{\sigma}_\alpha^2}{\chi^2_{\hat{\nu},0.025}}\right) = (9.596762, 2143.497)$$

with $\hat{\nu} = 1.786694$. So the corresponding confidence limits for σ_α is (3.10, 46.30). Similarly, we can give an confidence limits for σ^2_β as (4.782825, 30.20658) with $\hat{\nu} = 9.99$. Therefore, an confidence limits for σ_β is (2.19, 5.50).

We can use $(\text{SS(Picies)}/\chi^2_{60,0.975}, \text{SS(Picies)}/\chi^2_{60,0.025})$ as an exact confidence limits for σ^2. That is (3.361438,6.916697). So the confidence limits for σ is (1.83, 2.63).

From the confidence limits for $\sigma_\alpha, \sigma_\beta, \sigma$, we can say the largest part variation comes from differences between machines.

(b) The confidence interval for μ is

$$(\overline{y}_{\dots} - t_{2,0.975}\sqrt{\text{MS(Machine)}/75},\ \overline{y}_{\dots} + t_{2,0.975}\sqrt{\text{MS(Machine)}/75}).$$

That is (19.42, 50.58).

3. The model fitted by lmer() in R gives the following estimates

```
Random effects:
 Groups   Name        Variance   Std.Dev.
 A        (Intercept) 1.1575e+01 3.4022e+00
 B        (Intercept) 3.6282e-09 6.0235e-05
 Residual             7.2564e+00 2.6938e+00
Fixed effects:
            Estimate Std. Error t value
(Intercept)   10.899      2.118   5.146
```

Use MCMC method to construct confidence intervals. The 95% confidence limits for μ as (2.283360, 18.532268), for σ as (1.8239391, 5.227488), for σ_α as (0.2677417, 22.641538), and for σ_β as (0.1657626, 13.375462).

4. (a)

```
Analysis of Variance Table
Response: y
          Df  Sum Sq Mean Sq F value  Pr(>F)
A          1  0.5445  0.5445  0.2598 0.62129
B          4  2.6920  0.6730  0.3212 0.85751
A:B        4 24.4980  6.1245  2.9227 0.07695 .
Residuals 10 20.9550  2.0955
```

(b) The 95% confidence limits for σ^2 is $(SSE/\chi^2_{10,0.975}, SSE/\chi^2_{10,0.025})$, which results (1.023035, 6.453704).Thus the confidence limits for σ is (1.01, 2.54).

(c) From the EMS in the ANOVA table, we know that $S^2 := (\text{MSAB}+\text{MSB})/4 - \text{MSE}/2$ is an estimate of $\sigma^2_\beta + \sigma^2_{\alpha\beta}$. Then the Cochran-Satterthwaite confidence limits for $\sigma^2_\beta + \sigma^2_{\alpha\beta}$ is

$$(\hat{\nu}S^2/\chi^2_{\hat{\nu},0.975},\ \hat{\nu}S^2/\chi^2_{\hat{\nu},0.025}) = (0.1031844,\ 56810.76),$$

where $\hat{\nu} = 0.6040552$. Then the confidence limits for the standard deviation is (0.32, 238.35).

(d) The REML gives us the following estimates for the variance components.

```
Random effects:
 Groups   Name        Variance   Std.Dev.
```

```
 A:B      (Intercept) 4.9312e-01 7.0222e-01
 B        (Intercept) 1.0477e-09 3.2369e-05
 A        (Intercept) 1.0477e-09 3.2369e-05
 Residual             2.0955e+00 1.4476e+00
```

Thus, the repeatability and reproducibility standard deviation based on REML is 1.4476 and 0.7022. Based on the confidence limits for $\sigma^2_{\alpha\beta}$ and σ^2_β using MCMC, we can calculate the confidence interval for $\sqrt{\sigma^2_\beta + \sigma^2_{\alpha\beta}}$ as (0.1833, 3.8518).

5. (a) The following is the output from the fixed effects model.

```
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   9.5929     0.1040  92.245 2.13e-13 ***
A1            1.2849     0.1040  12.356 1.72e-06 ***
B1           -0.2476     0.1473  -1.681    0.131
B2            1.5571     0.1541  10.104 7.86e-06 ***
FBlock1      -1.0286     0.1053  -9.768 1.01e-05 ***
A1:B1         0.2603     0.1487   1.751    0.118
A1:B2        -0.2349     0.1541  -1.524    0.166

Residual standard error: 0.394 on 8 degrees of freedom Multiple
R-Squared: 0.9793,     Adjusted R-squared: 0.9638 F-statistic:
63.07 on 6 and 8 DF,  p-value: 2.665e-06
```

Thus an estimate of the σ is 0.394.

(b) The following output is obtained by treating Block as random effect. We can see that the influence on the estimates of the fixed effects are small. The estimate of σ in this model is 0.39402, which is almost the same as the estimate in the fixed block model.

```
Random effects:
 Groups   Name        Variance Std.Dev.
 FBlock   (Intercept) 2.09247  1.44654
 Residual             0.15525  0.39402
number of obs: 15, groups: FBlock, 2
Fixed effects:
            Estimate Std. Error t value
(Intercept)   9.5935     1.0281   9.331
A1            1.2843     0.1040  12.350
B1           -0.2464     0.1473  -1.673
B2            1.5565     0.1541  10.100
A1:B1         0.2627     0.1487   1.767
A1:B2        -0.2343     0.1541  -1.520
```

A 95% confidence interval for σ in fixed effects model is (0.266, 0.755), which is calculated by $\left(\sqrt{SSE/\chi^2_{8,0.975}}, \sqrt{SSE/\chi^2_{8,0.025}}\right)$. For the computation of confidence interval in the random block model, we call MCMC here. It gives a confidence interval as (0.25, 0.73). From the fixed effect model, we can calculate that $(\tau_2^2 + \tau_1^2)$ is 2.116036, which is almost the same with the estimate of the σ_τ in the random block model.

(c) The BLUE and BLUP in fixed effects model for $\mu + \alpha_1 + \beta_1 + \alpha\beta_{11} + \tau_1$ are both 9.861905. The BLUP in random effects model for $\mu + \alpha_1 + \beta_1 + \alpha\beta_{11} + \tau_1$ is 9.876288. I will use $9.5935 + 1.2843 - 0.2464 + 0.2627 = 10.8941$ to predict the value $\mu + \alpha_1 + \beta_1 + \alpha\beta_{11} + \tau_3$. In the fixed effect model, τ_3 does not make sense, since we assume the block as fixed effect.

6. (a) The first ANOVA table from the lmf1.out is

```
       Analysis of Variance Table
             Df  Sum Sq Mean Sq   Corresponds
TIME          2   67073   33537     SSA
AD            1  168151  168151     SSB
TIME:AD       2     391     196     SSAB
AD:TEST       8 1833681  229210     SSC(B)
TIME:AD:TEST 16    5727     358     SSE
Residuals     0       0
```

The second ANOVA table from the lmf2.out is

```
      Analysis of Variance Table

             Df  Sum Sq Mean Sq   Corresponds
TIME          2   67073   33537     SSA
AD            1  168151  168151     SSB
TEST          4  397490   99373     SSC
TIME:AD       2     391     196     SSAB
TIME:TEST     8    2475     309     SSAC
AD:TEST       4 1436191  359048     SSBC
TIME:AD:TEST  8    3252     407     SSABC
Residuals     0       0
```

We see that SSC(B)=SSC+SSBC=397490+1436191=1833681 and SSE=SSAC+SSABC=2475+3252=5727. The other SS are the same with the results in the above table.

(b) The MCMC 95% confidence limits for σ_γ and σ are (182.5583, 508.7239) and (13.80992, 28.45042) respectively.

(c) The confidence interval for σ is (14.09048, 28.79373). An estimate of σ_γ^2 is $\hat{\sigma}_\gamma^2 = \frac{1}{3}(MSC(B) - MSE) = \frac{1}{3}(229210 - 358) = 76284$. Thus a Cochran-Satterthwaite approximate degree of freedom of chi-square distribution is $\hat{\nu} = 7.975$. Thus a confidence limits for σ_γ is (186.467, 529.839), which is a little wider than that in part (b).

(d) A 95% confidence interval for the difference in Time 1 and Time 2 main effects is

$$(y_{1..} - y_{2..}) \pm t_{16,\,0.975}\sqrt{\frac{2MSE}{10}} = (-98.4, -62.5).$$

A 95% confidence interval for the difference in Ad Campaign 1 and Ad Campaign 2 main effects is

$$(y_{.1.} - y_{.2.}) \pm t_{8,\,0.975}\sqrt{\frac{2MSC(B)}{15}} = (-253.4, 552.9).$$

(e) Using the following estimates of the fixed effects, we can predict the Time 2 sales result for Ad Campaign 2 in another (untested) market as $\mu + \alpha_2 + \beta_2 + \alpha\beta_{22} = 653.4$.

```
Fixed effects:
            Estimate Std. Error t value
(Intercept) 664.5333    87.3770   7.605
```

```
TIME1       -16.1333     4.8860  -3.302
TIME2        64.2667     4.8860  13.153
AD1          74.8667    87.3770   0.857
TIME1:AD1    -4.6667     4.8860  -0.955
TIME2:AD1     0.5333     4.8860   0.109
```