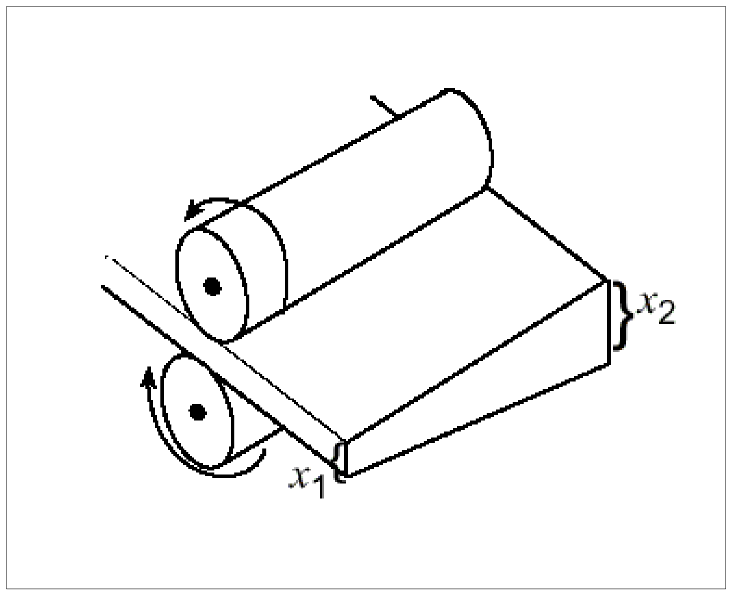
Basic Multivariate Process Monitoring

(Section 4.3 of Vardeman and Jobe)

Realm of Application

- Where *relationships between variables* matter



- Where there will be "automatic" calculation

“Obvious” Points About Such Situations

- Separate monitoring of variables (e.g. using multiple $\overline{x}$ charts) can never hope to detect “unusual relationships” … a single summary of all variables is needed
- Practical use of an “out of control” signal will require digging into the data to track down *what* has changed
- One must have (and use) some measure of “relationship” between variables

Correlation (A Measure of Linear Association Between *x* and *y*)

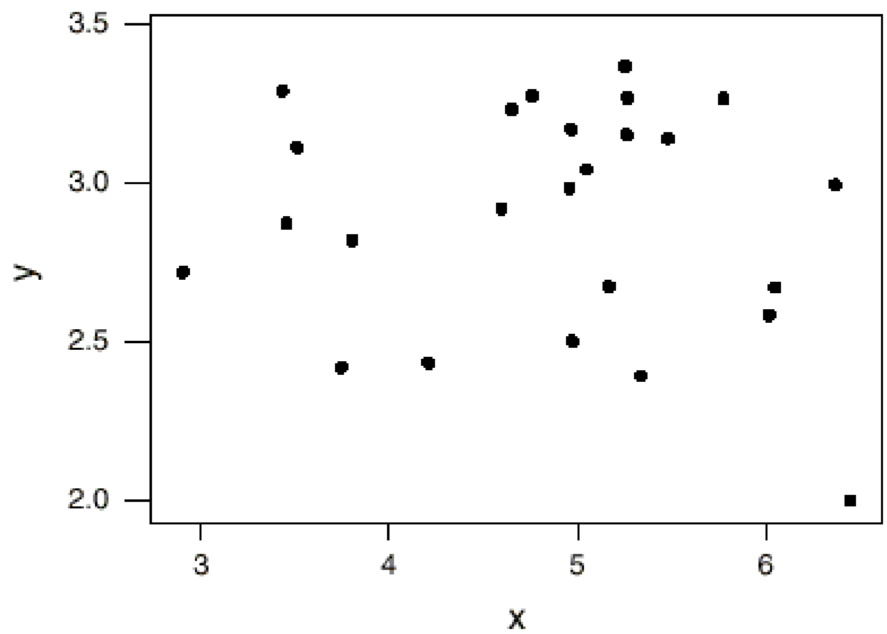
- Sample version is

$$r_{xy} = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\left(\sum (x-\bar{x})^2\right)\left(\sum (y-\bar{y})^2\right)}}$$

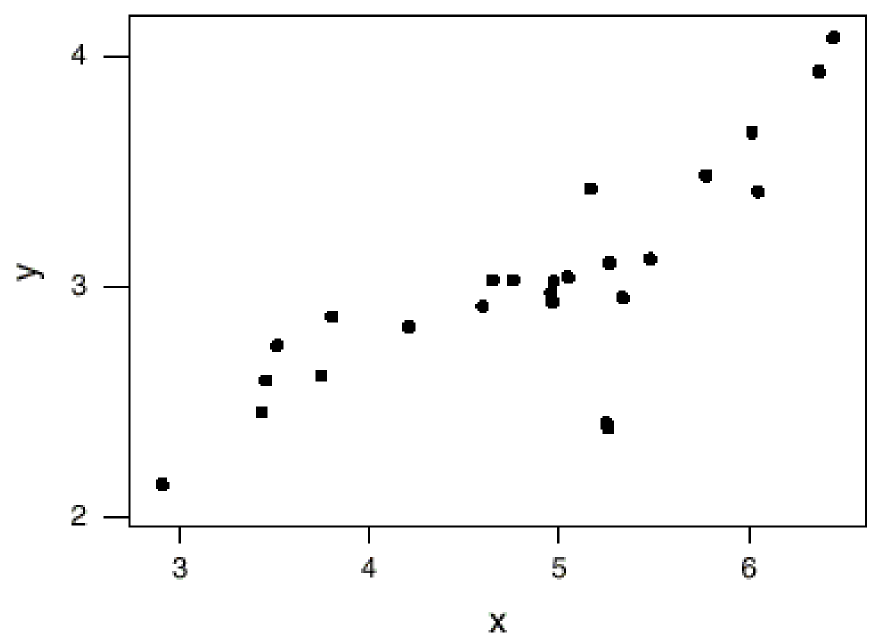
and $-1 \le r_{xy} \le 1$

- We'll call the "population"/theoretical version r_{xy} and $-1 \le r_{xy} \le 1$

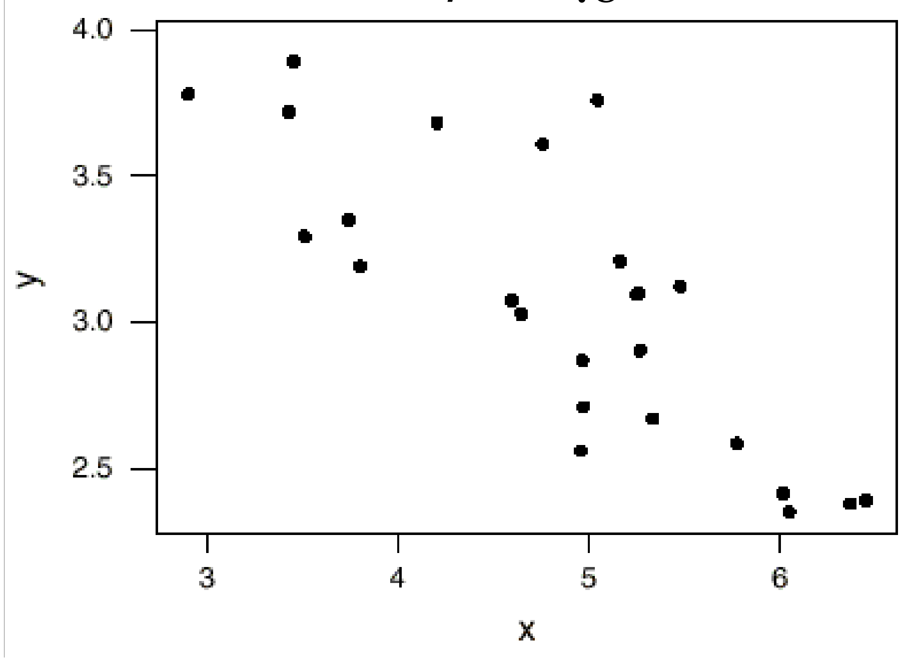
$r \approx -.1$



$r \approx .8$



$r \approx -.8$



Use of Correlation ("Standard Relationship") Information for *p* Variables??

- Collect correlations and variances in a matrix

$$\underset{p\times p}{V} = \begin{bmatrix} s_1^2 & r_{12}s_1 s_2 & \cdots & r_{1p}s_1 s_p \\ r_{21}s_1 s_2 & s_2^2 & \cdots & r_{2p}s_2 s_p \\ \vdots & \vdots & \ddots & \vdots \\ r_{p1}s_1 s_p & r_{p2}s_2 s_p & \cdots & s_p^2 \end{bmatrix}$$

- Plot the summary statistic (4.24)

$$X^2 = n\left(\underset{p\times 1}{\overline{x}} - \underset{p\times 1}{m}\right)' \underset{p\times p}{V}^{-1}\left(\underset{p\times 1}{\overline{x}} - \underset{p\times 1}{m}\right)$$

More Use of Correlation Information

- Use an upper control limit (only)

$$CL_{X^2} = p$$

$$UCL_{X^2} = p + 3\sqrt{2p}$$

- Large values of the plotted statistic signal
  - a change in one of the *p* means or standard deviations, OR
  - a change in the relationship between at least 2 of the *p* variables

??? Interpretation ???

- $p=2$ version of this can be written without matrix notation

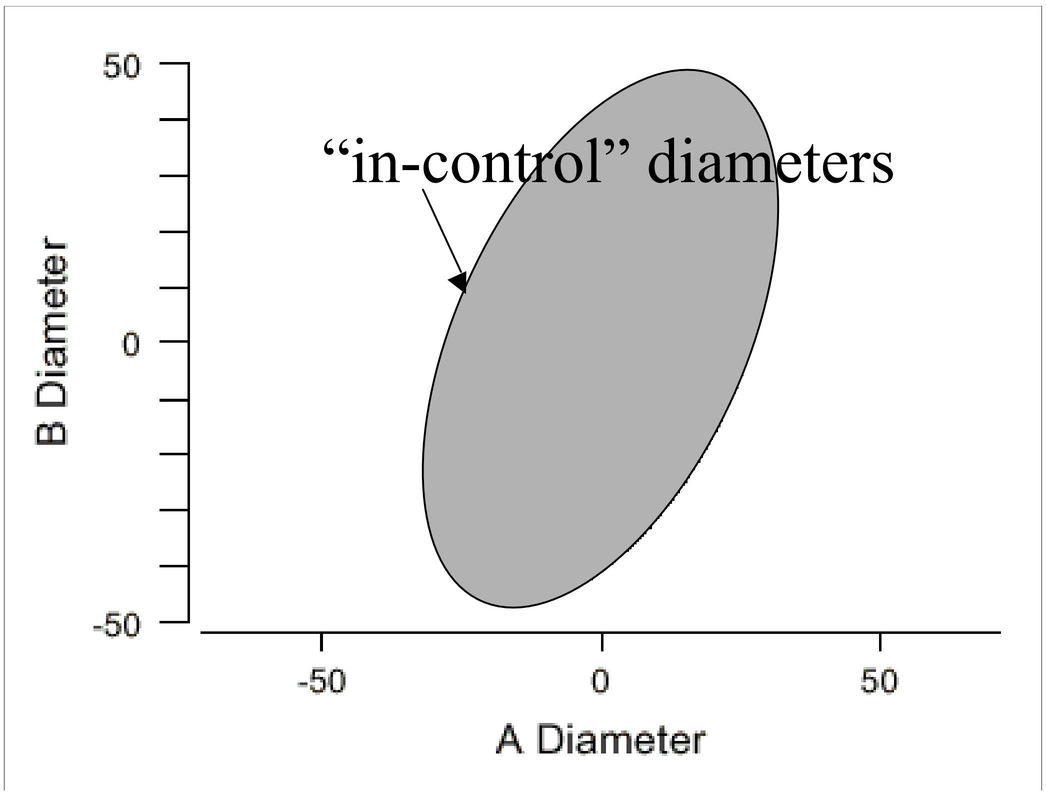
$$X^2 = \frac{1}{(1-r_{12}^2)}\left(\left(\frac{\bar{x}_1 - m_1}{s_1/\sqrt{n}}\right)^2 - 2r_{12}\left(\frac{\bar{x}_1 - m_1}{s_1/\sqrt{n}}\right)\left(\frac{\bar{x}_2 - m_2}{s_2/\sqrt{n}}\right) + \left(\frac{\bar{x}_2 - m_2}{s_2/\sqrt{n}}\right)^2\right)$$

- in-control values of this statistic come from vectors of sample means inside appropriate p-dimensional "footballs" centered at the vector of standard means with size and orientation controlled by V

Example 4.5

$$n = 1 \quad p = 2 \quad \underset{2\times 1}{m} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad s_1 = 15 \quad s_2 = 18 \quad r_{12} = .6$$

$$X^2 = \frac{x_1^2}{92.16} - \frac{x_1 x_2}{144} + \frac{x_2^2}{207.36}$$



Retrospective Analyses

- Requires the same kind of pooling together of information from *r* samples as for retrospective $\overline{x}$ charts
- For a single sample (l)

$$\underset{p\times p}{\widehat{V}} = \begin{bmatrix} s_1^2 & r_{12}s_1s_2 & \cdots & r_{1p}s_1s_p \\ r_{21}s_1s_2 & s_2^2 & \cdots & r_{2p}s_2s_p \\ \vdots & \vdots & \ddots & \vdots \\ r_{p1}s_1s_p & r_{p2}s_2s_p & \cdots & s_p^2 \end{bmatrix}$$

and

$$\underset{p\times 1}{\overline{x}} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_p \end{bmatrix}$$

Retrospective Analyses (cont'd)

- Sensible pooled estimators of $\underset{p\times 1}{m}$ and $\underset{p\times p}{V}$ are then

$$\underset{p\times 1}{\widehat{m}} = \left(\sum_{i=1}^{r} n_l \underset{p\times 1}{\overline{x}_l} \right) / \left(\sum_{i=1}^{r} n_l \right)$$

and

$$\underset{p\times p}{\widehat{V}} = \left(\sum_{i=1}^{r} (n_l - 1) \underset{p\times p}{\widehat{V}_l} \right) / \left(\sum_{i=1}^{r} (n_l - 1) \right)$$

(this is pooling of sample-by-sample information … not pooling of r samples)

Workshop Exercise

- Consider two (A diameter, B diameter) pairs for Example 4.5 (see slide 9):

  $$(25,30) \quad \text{and} \quad (25,-30)$$

  – plot them on slide 9

  – compute X^2 for both of them

  – determine whether either unit would cause an out-of-control signal for either diameter *alone* on an individuals chart (an $n=1$ $\overline{x}$ chart)