IE 361 Module 21

Design and Analysis of Experiments: Part 2

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Reading: Section 6.2, *Statistical Quality Assurance Methods for Engineers*

In this module we begin to consider what can be learned about the action of several factors on a response variable, based on r samples collected under different sets of process conditions, i.e. under different combinations of levels of those factors. We begin with the simplest such situation, where there are two factors of interest, and one has data from all of the $r = I \cdot J$ different combinations of I levels of Factor A and J levels of Factor B.

Basic Notation and Graphical and Numerical Summaries of Two-Way Factorial Data

We consider a situation where some Factor A has I levels, another Factor B has J levels, and samples of a response y are obtained under each combination of a level i of Factor A and a level j of Factor B. This results in what can be thought of as a two-way table of data

$$y_{ijk} = \text{the } k\text{th response at level } i \text{ of Factor A and level } j \text{ of Factor B}$$

illustrated in Figure 1. This layout is **complete** in the sense that there are data in every cell. The terminology **factorial** used in the name of this section means that the combinations of levels of the two factors are considered, and the jargon "$I \times J$ factorial" (naming the number of levels of each factor) is common.

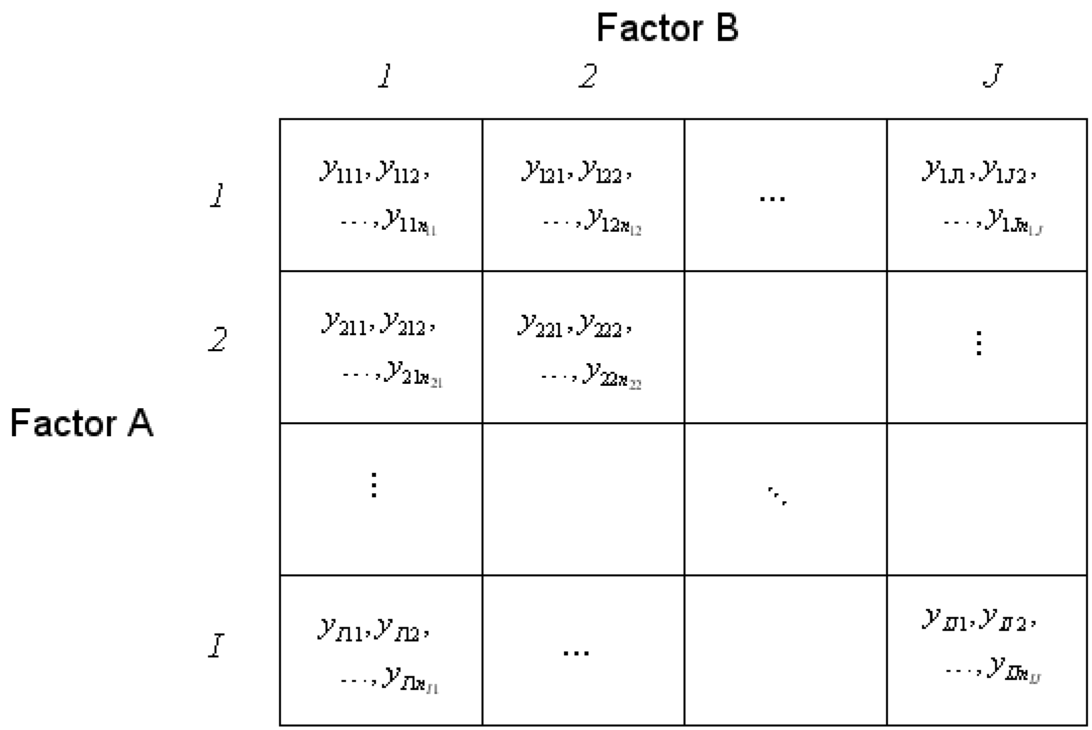

Figure 1: Two-Way Complete Factorial Data Layout

Example 21-1 The glass-phosphor study of Module 20 has 2×3 factorial structure. We repeat the summary statistics from Module 20, this time naming tube types not "1 through 6" but emphasizing their natural two-way structure through the use of double subscripts indicating glass (row) and phosphor (column) in the table (units are μA).

		Phosphor 1	Phosphor 2	Phosphor 3	
Glass	1	$\bar{y}_{11} = 285$ $s_{11}^2 = 25$	$\bar{y}_{12} = 301.67$ $s_{12}^2 = 58.33$	$\bar{y}_{13} = 281.67$ $s_{13}^2 = 108.33$	$\bar{y}_{1.} = 289.44$
	2	$\bar{y}_{21} = 235$ $s_{21}^2 = 25$	$\bar{y}_{22} = 245$ $s_{22}^2 = 175$	$\bar{y}_{23} = 225$ $s_{23}^2 = 25$	$\bar{y}_{2.} = 235$
		$\bar{y}_{.1} = 260$	$\bar{y}_{.2} = 273.33$	$\bar{y}_{.3} = 253.33$	$\bar{y}_{..} = 262.22$

Notice that the table contains not only cell sample means $\bar{y}_{ij}$ but also row and column averages of these, $\bar{y}_{i.}$'s and $\bar{y}_{.j}$'s respectively. These will prove useful for defining important summaries of two-way factorials in the balance of this section.

An effective way of portraying the results of a two-way factorial study is to plot sample means versus level of one factor, connecting plotted points for a given level of the second factor with line segments. Such a plot is commonly called an **interaction plot**. Figure 2 shows such a plot (with the raw data values also plotted on the figure).

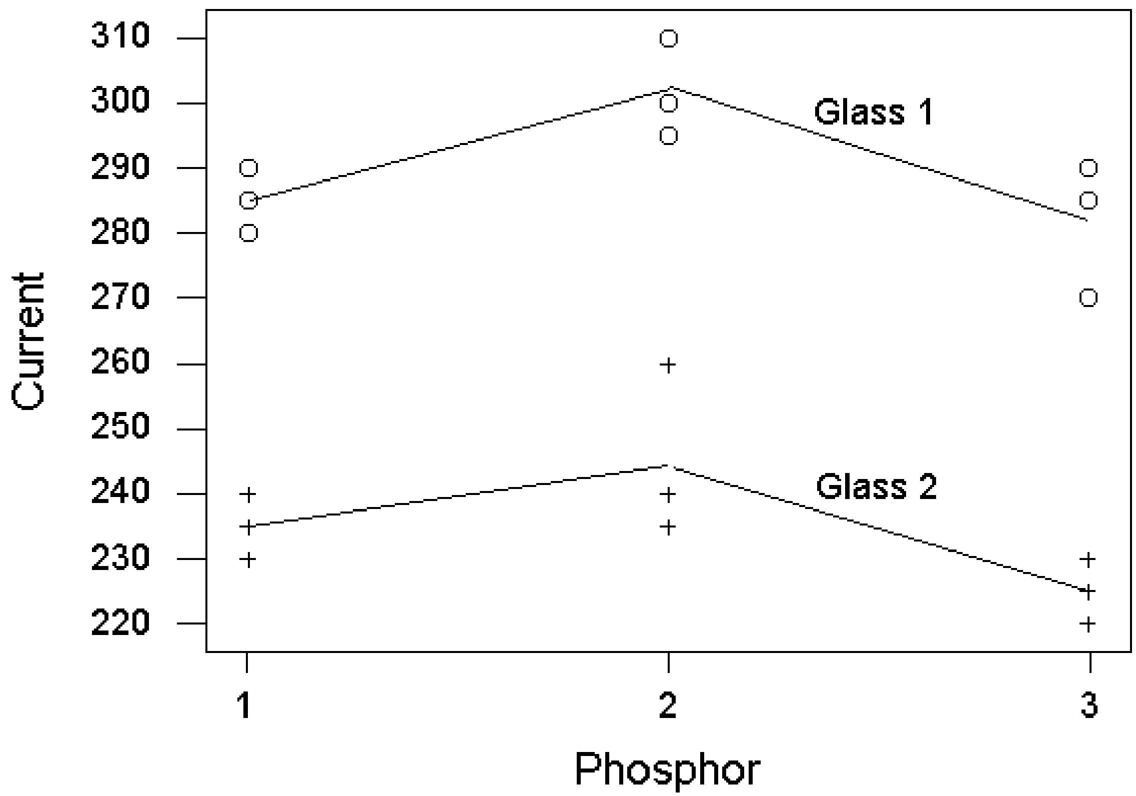

Figure 2: Interaction Plot for the Glass-Phosphor Study (With Raw Data Plotted on It)

An enhanced version of this kind of plot can be made by placing "error bars" around the plotted means. In Module 20 we saw that 95% confidence limits for the tube type means are of the form (using the double subscript notation of this module)

$$\bar{y}_{ij} \pm 10.44$$

Figure 3 is an interaction plot for the glass-phosphor study enhanced with error bars indicating the precision with which the individual current requirement means are known.

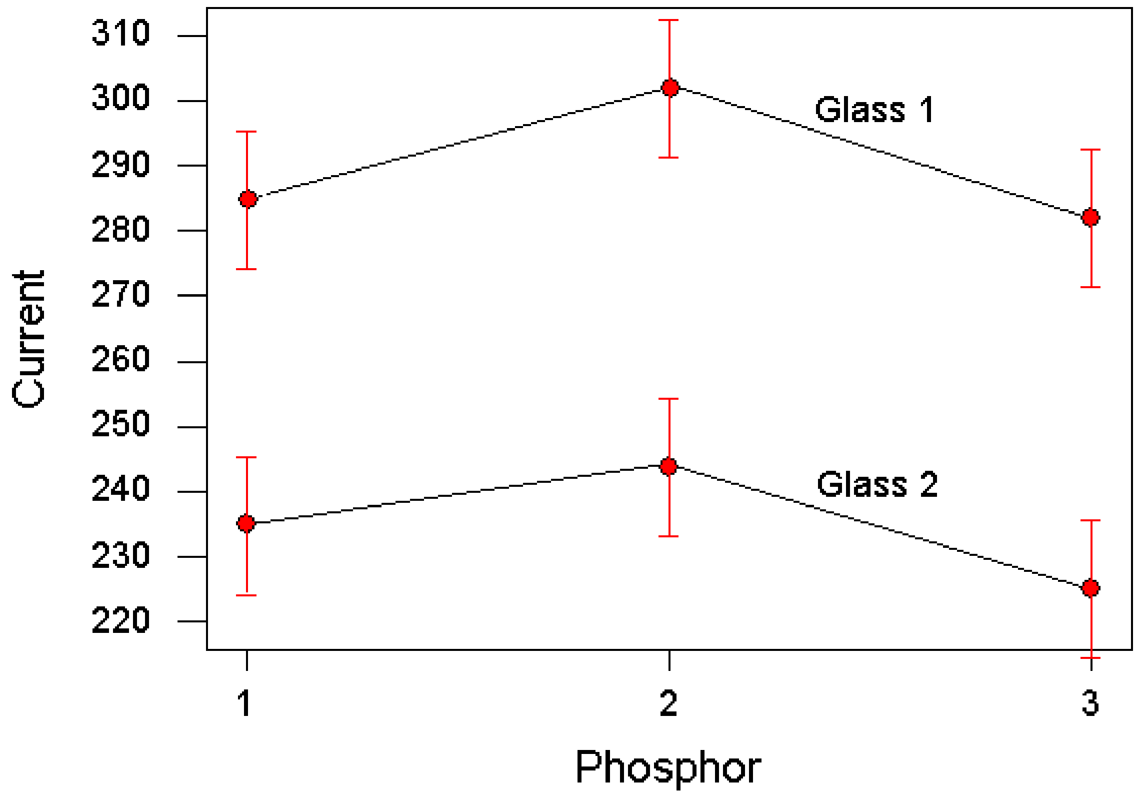

Figure 3: Interaction Plot for the Glass-Phosphor Study Enhanced With Error Bars From 95% Confidence Limits

The qualitative story told in Figure 3 is that:

- Glass 1 current requirements are clearly higher than those for Glass 2
- Phosphor 2 current requirements seem to be somewhat higher than those for Phosphors 1 and 3
- the "Glass" differences in current requirements seem to be larger than "Phosphor" differences
- the pattern in current requirement means for Glass 1 is similar to the pattern in current requirement means for Glass 2

A way to quantify qualitative insights like those above is to define so-called factorial effects. To begin, so called **main effects** of the factors are defined as appropriate row or column average $\bar{y}$'s minus the grand average $\bar{y}$. That is

$$\begin{aligned} a_i &= \bar{y}_{i.} - \bar{y}_{..} \\ &= \text{(the row } i \text{ average } \bar{y}) - \text{(the grand average } \bar{y}) \\ &= \text{the (fitted) main effect of the } i\text{th level of Factor A} \end{aligned}$$

and

$$\begin{aligned} b_j &= \bar{y}_{.j} - \bar{y}_{..} \\ &= \text{(the column } j \text{ average } \bar{y}) - \text{(the grand average } \bar{y}) \\ &= \text{the (fitted) main effect of the } j\text{th level of Factor B} \end{aligned}$$

Example 21-1 continued Using the row and column averages of current requirement sample means, we have

$$a_1 = 289.44 - 262.22 = 27.22 \text{ and } a_2 = 235 - 262.22 = -27.22$$

and

$$
\begin{aligned}
b_1 &= 260 - 262.22 - -2.22 \text{ and } b_2 = 273.33 - 262.22 = 11.11 \\
\text{and } b_3 &= 253.33 - 262.22 = -8.88
\end{aligned}
$$

The fact that the Factor A main effects are larger in absolute value than the Factor B main effects is consistent with the fact that the gap between the top and bottom profiles on Figure 3 is more pronounced than the up-then-down pattern seen on them. The fact that $a_1 > 0$ indicates that current requirements for Glass 1 are larger than for Glass 2 (that has $a_2 < 0$). (Similarly, the fact that $b_2 > 0$ indicates that current requirements for Phosphor 2 are larger than for Phosphors 1 and 3 that have $b_1 < 0$ and $b_3 < 0$.)

It is no accident that in the glass-phosphor example the (2) Factor A main effects add to 0 and the (3) Factor B main effects also add to 0. This is an algebraic consequence of the definitions of these quantities and can be used as a check on one's calculations.

In some cases the fitted main effects in a two-way factorial essentially capture the entire story told in the data set, in the sense that for each combination of a level i of Factor A and a level j of Factor B

$$\bar{y}_{ij} \approx \bar{y}_{..} + a_i + b_j$$

(the sample means can essentially be reconstructed from an overall mean and Factor A and Factor B main effects). The glass-phosphor example is such a case.

Example 21-1 continued The following two tables can be used to compare the glass-phosphor means $\bar{y}_{ij}$ and the quantities $\bar{y}_{..} + a_i + b_j$.

Table of $\bar{y}_{ij}$'s

		Phosphor 1	2	3
Glass	1	$\bar{y}_{11} = 285$	$\bar{y}_{12} = 301.67$	$\bar{y}_{13} = 281.67$
	2	$\bar{y}_{21} = 235$	$\bar{y}_{22} = 245$	$\bar{y}_{23} = 225$

Table of $\bar{y}_{..} + a_i + b_j$'s

		Phosphor 1	2	3
Glass	1	287.22	300.55	280.56
	2	232.78	246.11	226.11

For example,

$$\bar{y}_{..} + a_1 + b_1 = 262.22 + 27.22 + (-2.22) = 287.22$$

The fact that the two tables are very much alike is a reflection of the fact that the plots of means show fairly parallel traces. A way to measure *lack of parallelism* on a plot of means is to compute so called fitted **interactions**

$$\begin{aligned} ab_{ij} &= \bar{y}_{ij} - \left(\bar{y}_{..} + a_i + b_j\right) \\ &= \text{the difference between what is observed} \\ &\quad \text{at level } i \text{ of Factor A and level } j \text{ of Factor B} \\ &\quad \text{and what can be accounted for in terms of an} \\ &\quad \text{overall mean and the Factor A level } i \text{ main} \\ &\quad \text{effect and the Factor B level } j \text{ main effect} \end{aligned}$$

Example 21-1 continued The differences cell-by-cell of the entries in the two previous table are the fitted interactions ab_{ij} given in the table below.

Table of ab_{ij}'s

		Phosphor 1	Phosphor 2	Phosphor 3
Glass	1	-2.22	1.11	1.11
	2	2.22	-1.11	-1.11

For example,

$$ab_{11} = 285 - 287.22 = -2.22$$

The fitted interactions are smaller in absolute value than the fitted main effects of either Factor A or Factor B. Interactions measure lack of parallelism on an interaction plot, and their small size in the glass-phosphor example indicate that one can more or less think of the factors "Glass" and "Phosphor" as acting "separately" on the current requirement variable.

It is no accident that in the glass-phosphor example the $(2 \times 3 = 6)$ fitted interactions add to 0 across any row or down any column of the two way table (across any level of either factor). This is again an algebraic consequence of the definitions of these quantities.

Inference for Effects in Two-Way Factorial Studies

Before basing serious real world decisions on the perceived sizes of effects of experimental factors on a response, it is important to have some comfort that one is seeing more than is explainable as "just experimental variation." One must be reasonably certain that the effects one sees are more than just background noise. In the present two-way factorial context, that means that beyond computing fitted main effects and interactions a_i, b_j, and ab_{ij} one really needs some way of judging whether they are clearly "more than just noise." Attaching margins of error to them is a way of doing this, and the key to seeing how to find relevant margins of error is to recognize that fitted effects are linear combinations of $\bar{y}$'s, $\hat{L}$'s in the notation of Module 20.

Take, for example, the quantity b_2 in the 2×3 glass-phosphor study. This was

$$\begin{aligned} b_2 &= \bar{y}_{.2} - \bar{y}_{..} \\ &= \frac{1}{2}(\bar{y}_{12} + \bar{y}_{22}) \\ &\quad -\frac{1}{6}(\bar{y}_{11} + \bar{y}_{12} + \bar{y}_{13} + \bar{y}_{21} + \bar{y}_{22} + \bar{y}_{23}) \\ &= -\frac{1}{6}\bar{y}_{11} + \frac{1}{3}\bar{y}_{12} - \frac{1}{6}\bar{y}_{13} - \frac{1}{6}\bar{y}_{21} + \frac{1}{3}\bar{y}_{22} - \frac{1}{6}\bar{y}_{23} \end{aligned}$$

which is indeed of the form $\hat{L} = c_1\bar{y}_1 + c_2\bar{y}_2 + \cdots + c_r\bar{y}_r$.

The fact that fitted factorial effects are $\hat{L}$'s means that there are corresponding linear combinations of population/theoretical means μ (there are corresponding L's) and the methods of Module 20 can be used to make confidence limits based on the fitted effects ... find margins of error to attach to the fitted effects. That is we may use

$$\widehat{Effect} \pm t s_{\text{pooled}} \sqrt{\frac{c_{11}^2}{n_{11}} + \cdots + \frac{c_{IJ}^2}{n_{IJ}}}$$

to make confidence limits based on fitted effects $\widehat{Effect}$ if we can see what are the appropriate sums to put under the square root in the formula. And there are some fairly simple "hand calculation" formulas for what goes under the root in the case of two-way studies, particularly for cases where all $n_{ij} = m$ (there is a common fixed sample size). (Standard jargon is that this is the **balanced data** case.) Table 6.5 of *SQAME* gives the balanced data formulas and is essentially reproduced below. (Table 6.6 gives general formulas not requiring balanced data.)

L	$\hat{L}$	$\frac{c_{11}^2}{n_{11}} + \cdots + \frac{c_{IJ}^2}{n_{IJ}}$
$\alpha\beta_{ij}$	ab_{ij}	$\frac{(I-1)(J-1)}{mIJ}$
α_i	a_i	$\frac{I-1}{mIJ}$
$\alpha_i - \alpha_{i'}$	$a_i - a_{i'}$	$\frac{2}{mJ}$
β_j	b_j	$\frac{J-1}{mIJ}$
$\beta_j - \beta_{j'}$	$b_j - b_{j'}$	$\frac{2}{mI}$

Example 21-1 continued In the glass-phosphor study, $I = 2$, $J = 3$, and the common sample size is $m = 3$. So a margin of error to associate with any of the fitted interactions ab_{ij} based on 95% two-sided confidence limits is

$$ts_{\text{pooled}}\sqrt{\frac{(2-1)(3-1)}{3(2)(3)}} = 2.179(8.3)\sqrt{\frac{1}{9}}$$
$$= 6.0\,\mu\text{A}$$

and reviewing the table giving the fitted interactions, we see that all $I \times J = 6$ of them are smaller than this in absolute value. The lack of parallelism seen on the interaction plots is not only small in comparison to the size of fitted main effects, it is in fact "down in the noise range." This is quantitative confirmation of the story in this regard told on Figure 3.

A margin of error to be applied to the difference in fitted Factor A main effects $(\alpha_2 - \alpha_1 = (\bar{y}_{2.} - \bar{y}_{..}) - (\bar{y}_{1.} - \bar{y}_{..}) = \bar{y}_{2.} - \bar{y}_{1.} = -54.44)$ is then

$$ts_{\text{pooled}}\sqrt{\frac{2}{3\cdot 3}} = 2.179(8.3)\sqrt{\frac{2}{9}}$$
$$= 8.6\,\mu\text{A}$$

Since $|-54.44| > 8.6$ there is then a difference between the glass 1 and glass 2 main effects that is clearly more than experimental noise, again providing quantitative confirmation of what seems "obvious" on Figure 3.

And finally, a margin of error to be applied to any difference in fitted Factor B main effects $(\beta_j - \beta_{j'} = \left(\bar{y}_{.j} - \bar{y}_{..}\right) - \left(\bar{y}_{.j'} - \bar{y}_{..}\right) = \bar{y}_{.j} - \bar{y}_{j'})$ is then

$$\begin{aligned} ts_{\text{pooled}}\sqrt{\frac{2}{3 \cdot 2}} &= 2.179\,(8.3)\,\sqrt{\frac{1}{3}} \\ &= 10.5\,\mu\text{A} \end{aligned}$$

Looking again at the fitted phosphor main effects and their differences, the difference between phosphor 2 main effects and those of either of the other two phosphors is clearly more than experimental noise, but the observed difference between phosphor 1 and 3 main effects is "in the noise range." Once again, this is quantitative confirmation of what in retrospect seems "obvious" on Figure 3.

The virtue of inventing numerical measures like the main effects and interactions and learning how to attach margins of error to them over what can be seen on a plot like Figure 3, is that the numerical ideas generalize to many factors, where there is no obvious way to make pictures to allow us to "see" what is going on. In the next module we consider 3 (and higher) way factorials, placing primary attention on the case where every factor has just 2 levels, and make heavy use of such fitted factorial effects.