IE 361 Module 11

Shewhart Control Charts for Measurements

Prof.s Stephen B. Vardeman and Max D. Morris

Reading: Section 3.2, *Statistical Quality Assurance Methods for Engineers*

In this module we consider Shewhart control charts for measurements (or so called "variables data" in old time SQC jargon). As our featured example, we will use the data from an IE 361 Deming drama. These are recorded in Figure 1. (The "red bag" was an earlier version of the current "brown bag," i.e. had process parameters $\mu = 5$ and $\sigma = 1.715$ and was approximately normal.)

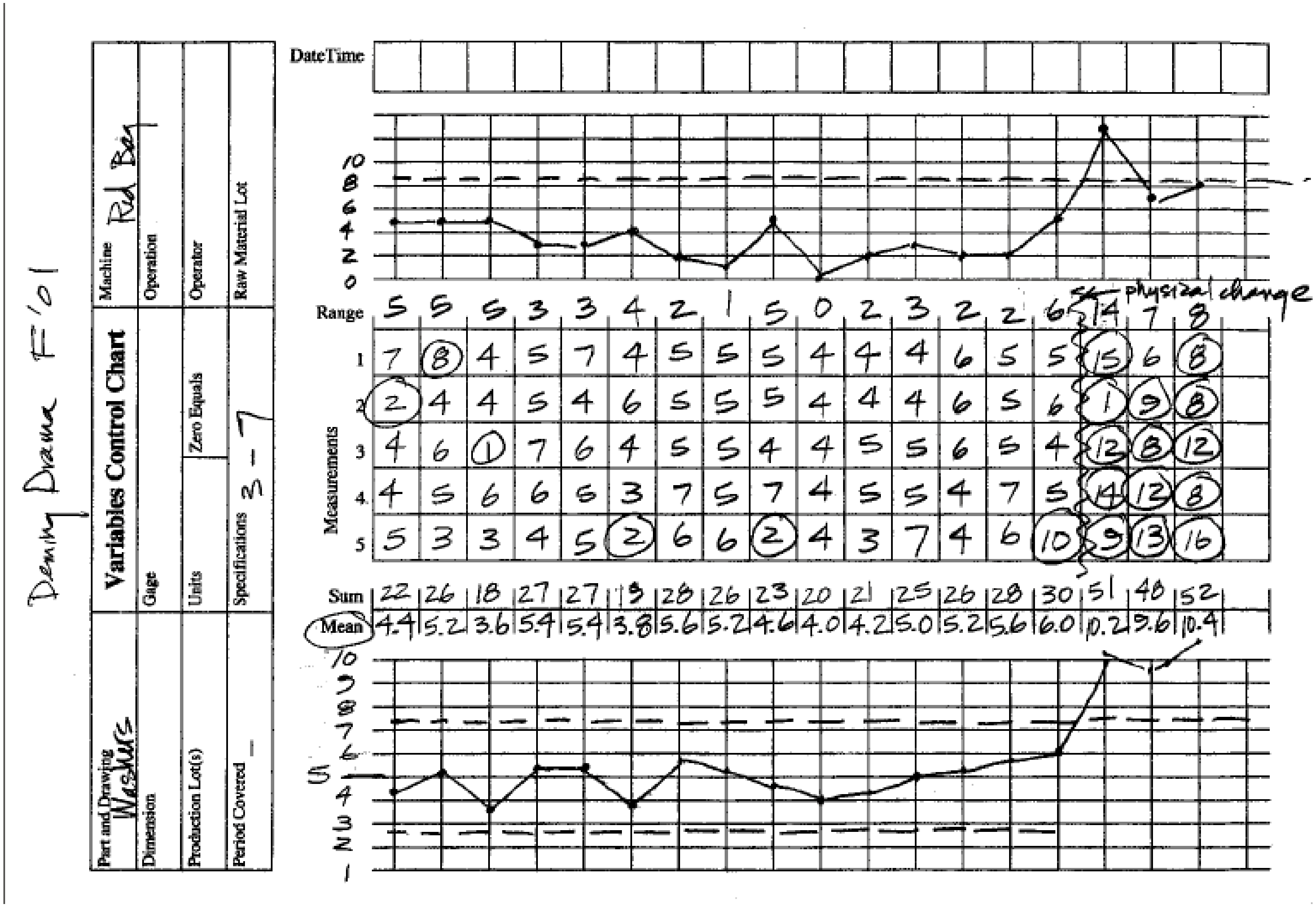

Figure 1: Data from an IE 361 Deming Drama

$\bar{x}$ Charts

We introduced the topic of Shewhart control charts in Module 10 using the most famous of all such charts, the $\bar{x}$ charts. To review, we saw that the (approximately) normal distribution of $\bar{x}$ (with mean $\mu_{\bar{x}} = \mu$ and $\sigma_{\bar{x}} = \sigma/\sqrt{n}$) leads to standards given control limits for $\bar{x}$

$$UCL_{\bar{x}} = \mu + 3\,\frac{\sigma}{\sqrt{n}} \qquad \text{and} \qquad LCL_{\bar{x}} = \mu - 3\,\frac{\sigma}{\sqrt{n}}$$

Further, we saw that in a retrospective situation like that illustrated in Figure 1 where sample means $\bar{x}$ and sample ranges R are computed, estimates

$$\hat{\mu} = \overline{\overline{x}} \;\text{ and }\; \hat{\sigma} = \bar{R}/d_2$$

can be substituted to produce retrospective control limits for $\bar{x}$

$$UCL_{\overline{x}} = \overline{\overline{x}} + 3\,\frac{\bar{R}}{d_2\sqrt{n}} \qquad \text{and} \qquad LCL_{\overline{x}} = \overline{\overline{x}} - 3\,\frac{\bar{R}}{d_2\sqrt{n}}$$

In fact, it is traditional to set

$$A_2 = \frac{3}{d_2\sqrt{n}}$$

and rewrite these retrospective control limits as

$$UCL_{\overline{x}} = \overline{\overline{x}} + A_2\bar{R} \qquad \text{and} \qquad LCL_{\overline{x}} = \overline{\overline{x}} - A_2\bar{R}$$

Example 11-1 We saw in Module 10 that (since for the brown bag $\mu = 5$ and $\sigma = 1.715$) standards given control limits for $\bar{x}$ are

$$UCL_{\overline{x}} = 5 + 3\,\frac{1.715}{\sqrt{5}} = 7.3 \qquad \text{and} \qquad LCL_{\overline{x}} = 5 - 3\,\frac{1.715}{\sqrt{5}} = 2.7$$

These limits are marked on the $\bar{x}$ control chart in Figure 1 and we can see that if they had been applied to $\bar{x}$'s in real time, process change would have been detected at sample 16.

The 18 sample means and ranges from Figure 1 average to

$$\overline{\overline{x}} = 5.744 \quad \text{and} \quad \bar{R} = 4.278$$

So retrospective limits for $\bar{x}$ are (since the sample size is $n = 5$)

$$\begin{aligned} UCL_{\overline{x}} &= 5.744 + .577\,(4.278) \\ &= 8.21 \end{aligned}$$

and

$$\begin{aligned} LCL_{\overline{x}} &= 5.744 - .577\,(4.278) \\ &= 3.28 \end{aligned}$$

When these limits are applied retrospectively to the 18 sample means, we see that the last 3 values are outside of these, and there is thus evidence of process instability in the data of Figure 1.

R Charts

It is traditional (not as it turns out best practice, but traditional) to use an R chart as a companion to an $\bar{x}$ chart. (An s chart to be discussed next is actually a better choice than an R chart, but historical precedent makes R charts continue to be common.) The $\bar{x}$ chart is primarily useful for monitoring process aim, while an R (or s) chart is primarily a tool for monitoring process spread or short term variation.

In order to identify appropriate control limits for R one needs to know some probability facts about R based on a sample of size n from a normal distribution. As it turns out, R has a (non-standard) probability distribution (not one you met in Stat 231) with mean proportional to the standard deviation of the sampled process. The constant of proportionality is the d_2 that we have used to turn ranges into estimates of standard deviations, that is

$$\mu_R = d_2\sigma$$

Further, the standard deviation of the probability distribution for R is proportional to the standard deviation of the sampled process. The constant of proportionality is called d_3. That is,

$$\sigma_R = d_3\sigma$$

Taken together, these probability facts about R produce standards given control limits for R

$$UCL_R = (d_2 + 3d_3)\sigma \qquad \text{and} \qquad LCL_R = (d_2 - 3d_3)\sigma$$

or, if one defines

$$D_2 = (d_2 + 3d_3) \text{ and } D_1 = (d_2 - 3d_3)$$

these standards given limits are

$$UCL_R = D_2\sigma \qquad \text{and} \qquad LCL_R = D_1\sigma$$

Further, in a retrospective situation like that illustrated in Figure 1 where R's are computed, the estimate

$$\hat{\sigma} = \bar{R}/d_2$$

can be substituted to produce retrospective control limits for R

$$UCL_R = D_2\bar{R}/d_2 \qquad \text{and} \qquad LCL_R = D_1\bar{R}/d_2$$

It is traditional to set

$$D_4 = \frac{D_2}{d_2} \qquad \text{and} \qquad D_3 = \frac{D_1}{d_2}$$

and rewrite these retrospective control limits as

$$UCL_R = D_4 \bar{R} \qquad \text{and} \qquad LCL_R = D_3 \bar{R}$$

Example 11-2 Since $\sigma = 1.715$ for the brown bag, a standards given upper control limit for R based on $n = 5$ is

$$UCL_R = 4.918\,(1.715) = 8.43$$

(No standards given lower control limit is typically used, because for a sample size of only $n = 5$, the difference $(d_2 - 3d_3)$ turns out to be negative.) This limit is marked on the R control chart in Figure 1 and we can see that if it had been applied to R's in real time, process change would have been detected at sample 16.

Recalling that the 18 sample means and ranges from Figure 1 average to $\bar{R} = 4.278$, a retrospective upper control limit for R is

$$UCL_R = 2.115\,(4.278) = 9.05$$

When this limit is applied retrospectively to the 18 sample ranges, we see that the 16th sample range plots "out of control" and there is evidence of process instability in data in Figure 1.

s Charts

s charts represent a superior alternative to R charts. At the price of requiring more than "by hand" calculation (sample standard deviations being more difficult to compute than sample ranges), they provide typically quicker detection of process changes. In order to identify appropriate control limits for s one needs to know some probability facts about s based on a sample of size n from a normal distribution. It is a fact mentioned in Stat 231 (that actually stands behind the standard confidence limits for σ) that $(n-1)\,s^2/\sigma^2$ has a χ^2 probability distribution. It turns out to follow from this fact that s has mean proportional to the standard deviation of the sampled process. The constant of proportionality is something called c_4. That is,

$$\mu_s = c_4\sigma$$

Further, the standard deviation of the random variable s is proportional to the standard deviation of the sampled process. The constant of proportionality is called c_5 that is

$$\sigma_s = c_5\sigma$$

Taken together, these probability facts about s produce standards given control limits

$$UCL_s = (c_4 + 3c_5)\sigma \qquad \text{and} \qquad LCL_s = (c_4 - 3c_5)\sigma$$

or, if one defines

$$B_6 = (c_4 + 3c_5) \text{ and } B_5 = (c_4 - 3c_5)$$

these standards given limits are

$$UCL_s = B_6\sigma \qquad \text{and} \qquad LCL_s = B_5\sigma$$

Further, in a retrospective situation like that illustrated in Figure 1 where s values (instead of R values) are computed, the estimate

$$\hat{\sigma} = \bar{s}/c_4$$

can be substituted to produce retrospective control limits for s

$$UCL_s = B_6\bar{s}/c_4 \qquad \text{and} \qquad LCL_s = B_5\bar{s}/c_4$$

It is traditional to set

$$B_4 = \frac{B_6}{c_4} \qquad \text{and} \qquad B_3 = \frac{B_5}{c_4}$$

and rewrite these retrospective control limits as

$$UCL_s = B_4\bar{s} \qquad \text{and} \qquad LCL_s = B_3\bar{s}$$

A final bit of development concerning these retrospective s-based calculations is this. Using $\hat{\sigma} = \bar{s}/c_4$, possible retrospective $\bar{x}$ chart limits are

$$UCL_{\overline{x}} = \overline{\overline{x}} + 3\,\frac{\bar{s}}{c_4\sqrt{n}} \qquad \text{and} \qquad LCL_{\overline{x}} = \overline{\overline{x}} - 3\,\frac{\bar{s}}{c_4\sqrt{n}}$$

and it is traditional to set

$$A_3 = \frac{\bar{s}}{c_4\sqrt{n}}$$

and rewrite these retrospective control limits as

$$UCL_{\overline{x}} = \overline{\overline{x}} + A_3\bar{s} \qquad \text{and} \qquad LCL_{\overline{x}} = \overline{\overline{x}} - A_3\bar{s}$$

Example 11-3 Since $\sigma = 1.715$ for the brown bag, a standards given upper control limit for s is

$$UCL_s = 1.964\,(1.715) = 3.37$$

(No standards given lower control limit is typically used, because for a sample size of only $n = 5$, the difference $(c_4 - 3c_5)$ turns out to be negative.)

The following table shows the 18 sample standard deviations corresponding to the data in Figure 1.

Sample	1	2	3	4	5	6	7	8	9	10
s	1.82	1.92	1.82	1.14	1.14	1.48	.89	.45	1.82	0

	11	12	13	14	15	16	17	18
	.84	1.22	1.10	.89	2.35	5.63	2.88	3.58

It is evident from these s values that if the standards given control limit had been applied to s's in real time, process change would have been detected at sample 16.

The 18 values s average to $\bar{s} = 30.97/18 = 1.72$. So a retrospective upper control limit for s for the data of Figure 1 is

$$UCL_s = 2.089\,(1.72) = 3.59$$

and when this limit is applied retrospectively to the 18 sample standard deviations, we see that the 16th plots "out of control" and there is evidence of process instability in data of Figure 1.

Further, consider making retrospective control limits for $\bar{x}$ based on sample standard deviations. These are

$$\begin{aligned} UCL_{\overline{x}} &= 5.774 + 1.427\,(1.72) \\ &= 8.23 \end{aligned}$$

and

$$LCL_{\overline{x}} = 5.774 + 1.427\,(1.72)$$
$$= 3.32$$

When these limits are applied retrospectively to the 18 sample means from Figure 1, we see that the last 3 values are outside of these, and there is evidence of process instability in the data.

Median Charts

A computationally simpler (but not often used) alternative to the Shewhart $\bar{x}$ chart is Shewhart median ($\tilde{x}$) chart. Finding a median requires only putting a data set in order smallest to largest and then finding the middle value, $\tilde{x}$. This is a measure of process aim like the mean. But is it generally not as reliable as the mean. Differently put, it generally takes longer to detect process change using medians than using means. But in some rare contexts, computational simplicity may outweigh this lack of sensitivity.

In order to identify appropriate control limits for $\tilde{x}$ one needs to know some probability facts about $\tilde{x}$ based on a sample of size n from a normal distribution. As it turns out, $\tilde{x}$ has a (non-standard) probability distribution (not one you met in Stat 231) with mean equal to the process mean and standard deviation larger than that of $\bar{x}$ by a multiplicative factor that we will call κ. That is

$$\mu_{\tilde{x}} = \mu \text{ and } \sigma_{\tilde{x}} = \kappa \frac{\sigma}{\sqrt{n}}$$

where a small table of values for κ is given on page 72 of *SQAME*. These facts suggest standards given control limits for $\tilde{x}$

$$UCL_{\tilde{x}} = \mu + 3\kappa \frac{\sigma}{\sqrt{n}} \qquad \text{and} \qquad LCL_{\tilde{x}} = \mu - 3\kappa \frac{\sigma}{\sqrt{n}}$$

(Any sensible estimates of μ and σ could further be used to make retrospective limits for $\tilde{x}$.)

Example 11-4 Since for the brown bag process $\mu = 5$ and $\sigma = 1.715$, standards given control limits for $\tilde{x}$ based on $n = 5$ are

$$\begin{aligned} UCL_{\tilde{x}} &= 5 + 3\,(1.197)\,\frac{1.715}{\sqrt{5}} \\ &= 7.75 \end{aligned}$$

and

$$\begin{aligned} LCL_{\tilde{x}} &= 5 - 3\,(1.197)\,\frac{1.715}{\sqrt{5}} \\ &= 2.25 \end{aligned}$$

The 18 sample medians for the data of Figure 1 are as in the table below

Sample	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\tilde{x}$	4	5	4	5	5	4	5	5	5	4	4	5	6	5	5	12	9	8

So if the standards given control limits had been applied to $\tilde{x}$'s in real time, process change would have been detected at sample 16.

An Important Reminder!!!

It is worth saying again that control limits are NOT engineering specifications nor vice versa. In Module 10 we said and now say again that a process can be stable without being acceptable and vice versa. The table below again compares these two fundamentally different concepts.

Control Limits	Specifications
have to do with process stability	have to do with product acceptability
apply to Q	apply to individuals, x
usually derived from process data	derived from performance requirements