Solution to Final Exam

1. As \( n S_n \sim W_p(n, \Sigma) \), \( n S_n = \sum_{i=1}^{n} \frac{z_i^2}{c_i} \)

where \( \{z_i\}_{i=1}^{n} \sim N_p(0, \Sigma) \)

\[
E(\mathbf{tr}(nS_n^2)) = \mathbf{tr}E\left( \sum_{i=1}^{n} \frac{z_i^2}{c_i} \right)
\]

\[
= \mathbf{tr}\left\{ n E(3, \frac{3^t}{3}) + n(n-1) E(3, \frac{3^t}{3}, \frac{3^t}{3}) \right\}
\]

\[
= n E(3, \frac{3^t}{3}) + n(n-1) \mathbf{tr}(\Sigma)
\]

\( \text{Let} \quad \mathbf{z}_1 = \begin{pmatrix} \mathbf{z}_1^t \\ \mathbf{z}_1^p \end{pmatrix} \sim N_p(0, \Sigma) \quad \mathbf{z}_1^t = \sum \frac{1}{c_i} \begin{pmatrix} \mathbf{z}_1^t \\ \mathbf{z}_1^p \end{pmatrix} = \sum \mathbf{z}_1^t
\]

\( \text{Let} \quad \mathbf{z}_1 = \begin{pmatrix} \mathbf{z}_1^t \\ \mathbf{z}_1^p \end{pmatrix} \sim N_p(0, I) \)

\[
E(3, \frac{3^t}{3}) = E(\mathbf{z}_1, \Sigma \mathbf{z}_1) = E\left( \frac{1}{c_i} \mathbf{\sigma}_{ij}, \mathbf{z}_1^t \mathbf{z}_1^p \right)
\]

\[
= E\left( \frac{1}{c_i}, \mathbf{\sigma}_{ij}, \mathbf{\sigma}_{kl}, \mathbf{z}_1^t \mathbf{z}_1^p \right) = 3 \sum_{i=1}^p \mathbf{\sigma}_{ii}^2 + \sum_{i<j}^p \left[ \mathbf{\sigma}_{ij} \mathbf{\sigma}_{ij} + 2 \mathbf{\sigma}_{ij}^2 \right]
\]

\[
= \sum_{i=1}^p \mathbf{\sigma}_{ii}^2 + 2 \sum_{i<j}^p \mathbf{\sigma}_{ij}^2 = \mathbf{tr}(\Sigma) + 2 \mathbf{tr}(\Sigma^2)
\]

\[
\Rightarrow E(\mathbf{tr}(nS_n^2)) = n \mathbf{tr}(\Sigma) + n(n-1) \mathbf{tr}(\Sigma^2)
\]
\[ E(\tau^2(n, s_n)) = E\left( \sum_{i,j} 3_i' 3'_j \right) = n E(\beta_i^2) + n(n-1) + \tau^2(\Sigma) \]

corresponding to:
\[ \tau(\Sigma s_n) = \sum_i 3_i' 3_i \]

\[ = E(B_n^2) = \frac{1}{(n+2)(n-1)} \left[ E(n^2 \Sigma^2) - \frac{1}{n} E(\tau^2(\Sigma)) \right] \]

\[ = \frac{1}{(n+2)\sigma^2} \left[ n(n-1) + \tau^2(\Sigma^2) + (n-1) E(\beta_i^2) - (n-1) + \tau^2(\Sigma) \right] \]

\[ = \tau^2(\Sigma^2) \]

Q2.

\[ E(Q_n) = \frac{n_t + n_e}{n_s} \sum_{j=2}^{n_s} \frac{1}{\eta_j^2} \sum_{i, i_1, i_2, \ldots, i_4} E\left( 3_i' 3_i 3_{i_1}' 3_{i_2} 3_{i_3}' 3_{i_4} \right) \]

\[ = \frac{n_t + n_e}{n_s} \frac{1}{\eta_j^2} \left\{ \sum_{j=2}^{n_s} \frac{3-1}{2} E(3_i' 3_i 3_{i_1}) + \sum_{i_1 \neq i_3}^{n_s} \frac{3-1}{3} E(3_{i_1}' 3_{i_3} 3_{i_1}) \right\} \]

\[ + \frac{n_t + n_e}{n_s} \frac{1}{\eta_j^2} \sum_{i_1 \neq i_3}^{n_s} \left( \frac{3-1}{3} E(3_{i_1}' 3_{i_3} 3_{i_1}) \right)^2 + \frac{3-1}{3} E(3_i' 3_i 3_{i_1}) \]

\[ = \frac{n_t + n_e}{n_s} \frac{1}{\eta_j^2} \left\{ \frac{3-1}{2} E(3_i' 3_i 3_{i_1}) + 2 \sum_{i_1 \neq i_3}^{n_s} \frac{3-1}{3} \frac{E(3_{i_1}' 3_{i_3} 3_{i_1})^2}{\eta_{i_1}^2 \eta_{i_3}^2} + \frac{3-1}{3} \frac{E(3_i' 3_i 3_{i_1})}{\eta_{i_1}^2 \eta_{i_3}^2} \right\} \]
without abusing a notation too much, let

\[ X_j = \eta_j' Y_j \quad j = 1, \ldots, n_1 + n_2. \]

If \( i_1 \neq i_2 \), let \( P' \Sigma P = \tilde{\Sigma} = (\tilde{\sigma}_{ij})_{m \times m} \)

\[ E(X_{i_1} X_{i_2}) = E(\tilde{z}_{i_1} P' \Sigma P \tilde{z}_{i_2} \tilde{z}_{i_1} P' \Sigma P \tilde{z}_{i_2}) \]

\[ = \tilde{E} \left\{ \sum_{e_1, k_1}^{m} \sum_{e_2, k_2}^{m} \tilde{\sigma}_{i_1, k_1} \tilde{\sigma}_{i_2, k_2} \tilde{z}_{i_1, e_1} \tilde{z}_{i_2, e_2} \tilde{z}_{i_1, k_1} \tilde{z}_{i_2, k_2} \right\} \quad (1) \]

\[ = \sum_{e_1, k_1}^{m} \sum_{e_2, k_2}^{m} \tilde{\sigma}_{i_1, k_1} \tilde{\sigma}_{i_2, k_2} \tilde{E}(\tilde{z}_{i_1, e_1} \tilde{z}_{i_2, e_2}) \tilde{E}(\tilde{z}_{i_1, k_1} \tilde{z}_{i_2, k_2}) \]

\[ = \sum_{e_1, k_1}^{m} \sum_{e_2, k_2}^{m} \tilde{\sigma}_{i_1, k_1}^2 = \text{tr}(\tilde{\Sigma}^2) = \text{tr}(P' \Sigma P P' \Sigma P) = \text{tr}(\Sigma^2). \]

If \( i_1 = i_2 \) (i.e., direct cut from (1) above)

\[ = \tilde{E}(X_{i_1}^2) = \sum_{e_1, e_2, k_1, k_2}^{m} \tilde{\sigma}_{i_1, k_1} \tilde{\sigma}_{i_2, k_2} \tilde{E}(\tilde{z}_{i_1, e_1} \tilde{z}_{i_2, e_2} \tilde{z}_{i_1, k_1} \tilde{z}_{i_2, k_2}) \]

\[ = (3^4) \sum_{e_1, e_2}^{m} \tilde{\sigma}_{i_1, e_1}^2 + \frac{p}{p + k} \left\{ \tilde{\sigma}_{i_1, k_1} \tilde{\sigma}_{i_2, k_2}^2 + 2 \tilde{\sigma}_{i_1, k_1}^2 \right\} \]

\[ = \sum_{e_1, k_1}^{m} \tilde{\sigma}_{i_1, e_1} \tilde{\sigma}_{i_2, k_1} + 2 \sum_{e_1, k_1}^{m} \tilde{\sigma}_{i_1, k_1}^2 \quad (\because) = \text{tr}(\tilde{\Sigma}) + 2 \text{tr}(\tilde{\Sigma}^2) \]

\[ = \frac{2}{2} \left( \text{tr}(\tilde{\Sigma}^2) + 2 \text{tr}(\tilde{\Sigma}^4) \right) + \Delta \sum \tilde{\sigma}_{i_1, e_1}^2 \frac{\Delta}{2} \tilde{\sigma}_{i_1, k_1}^2 \]

\[ = o(\text{tr}(\tilde{\Sigma}^4)) \]
\[ E(Q_n) = \sum_{j=2}^{n+1} \frac{1}{n_j^{1/2}} \left\{ \sum_{i=1}^{j-1} \frac{2 \text{tr}(\Sigma^i) + \text{tr}^2(\Sigma^i)}{n_i^{1/4}} \right\} + o(\text{tr}(\Sigma^i)) \]

\[ + 2 \sum_{i \neq i_2} \frac{\text{tr}^2(\Sigma^i)}{n_{i_1}^{1/2} n_{i_2}^{1/2}} + \sum_{i \neq i_2} \frac{\text{tr}(\Sigma^i)}{n_{i_1}^{1/2} n_{i_2}^{1/2}} \]

Let
\[ A_{i_1 c} = \frac{2 \text{tr}(\Sigma^1) + \text{tr}^2(\Sigma)}{n_i^{1/4}} \leq 2 p \cdot \lambda_{\max} + O(n^2) \]
\[ = o(n^3) + O(n^3) = o(n^4). \]

\[ A_{i_2 c_1} = \frac{\text{tr}^2(\Sigma^1)}{n_{c_1}^{1/2} n_{c_2}^{1/2}} = O(n^{-2}) \]

\[ A_{i_2 c_2} = \frac{\text{tr}(\Sigma^i)}{n_{c_1}^{1/2} n_{c_2}^{1/2}} = o(n^4) \]

\[ \therefore E(Q_n) = \sum_{j=2}^{n+1} \frac{1}{n_j^{1/2}} \left\{ \sum_{i=1}^{j-1} \frac{2 \text{tr}(\Sigma^i) + \text{tr}^2(\Sigma^i)}{n_i^{1/4}} \right\} + o(\text{tr}(\Sigma^i)) \]

It can be shown that
\[ A_{i_1 c} = \max_{1 \leq i \leq n_{i_1}^{1/2}} A_{i_1 c} = o(n^4) \]
\[ A_2 = \max_{i,j} A_{i_2 c_2} = O(n^2) \]
\[ A_3 = \max_{i,j} A_{i_2 c_2} = o(n^4) \]
0 \leq E(Q_n) \leq \sum_{j=2}^{n_t+n_q} \frac{1}{n_j^{\gamma/2}} \left\{ (j-1) \cdot A_1 + 2 \frac{(j-1)(j-2)}{n_j^2} A_2 \right. \\
\left. + (j-1)(j-2) A_3 \right\} \\
= O\left( \frac{\sum_{j=2}^{n_t+n_q} (j-1)}{n_j^4} \cdot A_1 \right) + O\left( \frac{\sum_{j=2}^{n_t+n_q} (j-1)(j-2)}{n_j^2} A_2 \right) \\
+ O\left( \frac{\sum_{j=2}^{n_t+n_q} (j-1)(j-2)}{n_j^4} \cdot A_3 \right).

A_1 = \frac{\sum_{j=2}^{n_t+n_q} (j-1)}{2} = O(n^2)

A_2 = \frac{\sum_{j=2}^{n_t+n_q} (j-1)(j-2)}{2} = \left( \frac{\gamma^2 - 3j + 2}{6} \right) =

= \frac{(n_t+n_q)(n_t+n_q+1)(2n_t+2n_q+1)}{6} - \frac{3(n_t+n_q)(n_t+n_q-1)}{2} + 2(n_t+n_q)

= O(n^3)

\Rightarrow 0 \leq E(Q_n) \leq O\left( \frac{n^2 \gamma^{-1}}{n_j^4} \right) + O\left( \frac{n^3 \gamma^{-2}}{n_j^4} \right)

+ O\left( \frac{n^3 \gamma^{-4}}{n_j^4} \right) = \sum_{j=2}^{n_t+n_q} \frac{1}{n_j^{\gamma/2}} + O\left( \frac{n^2}{n_j^4} \right).