HOMEWORK 2

Experiment 1

$\mathbf{X}_1$, the model matrix corresponding to the assumed model, will be such that $\mathbf{X}_1'\mathbf{X} = N\mathbf{I}$. At each "corner", $\underline{x}' = (1, \underline{u}') = (1, \pm 1, \pm 1, ..., \pm 1)$. Hence at these points,

$$V(\hat{y}(\underline{u})) = \sigma^2 \underline{x}'(\mathbf{X}_1'\mathbf{X}_1)^{-1}\underline{x} = \sigma^2 \underline{x}'\underline{x}/N = \sigma^2(r+1)/N$$

$\mathbf{X}_2$, the model matrix corresponding to the excluded terms, will contain r columns of 1's (pure quadratic terms) and r-choose-2 columns of +1's and -1's. Further, because the fraction is of resolution $\geq$ IV, these columns will be such that:

$$\mathbf{X}_1'\mathbf{X}_2 = \begin{pmatrix} N & N & ... & N \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix}$$

(What would it have looked like if the design had been a regular fraction of resolution III?) Hence the alias matrix has the same form as $\mathbf{X}_1'\mathbf{X}_2$, except that the non-zero elements are 1's, and so

$$E(\hat{\underline{\theta}}_1) = \begin{pmatrix} \theta_0 + \sum_{i=1}^r \theta_{ii} \\ \theta_1 \\ \theta_2 \\ ... \\ \theta_r \end{pmatrix}, \quad E(\hat{y}(\underline{u})) = \theta_0 + \sum_{i=1}^r \theta_{ii} + \sum_{i=1}^r u_i\theta_i, \quad E(y(\underline{u})) - E(\hat{y}(\underline{u})) = \sum_{i<j=1}^r u_i u_j \theta_{ij}$$

Experiment 1.a

Now, θ_0 is absent from both the true and assumed models (since we're modeling $y - \theta_0$), and the first column of $\mathbf{X}_1$ is removed. Hence

$$V(\hat{y}(\underline{u})) = \sigma^2 r/N,$$

at least somewhat smaller than before. Because the first column of $\mathbf{X}_1$ is omitted, $\mathbf{A} = \mathbf{0}$, i.e. $\hat{\underline{\theta}}_1$ is unbiased even though the model is incorrect. So,

$$E(\hat{y}(\underline{u})) = \theta_0 + \sum_{i=1}^r u_i\theta_i, \quad E(y(\underline{u})) - E(\hat{y}(\underline{u})) = \sum_{i=1}^r \theta_{ii} + \sum_{i<j=1}^r u_i u_j \theta_{ij}.$$

Note that in this case, the response estimate *may* be *more* biased because $E(\hat{\underline{\theta}}_1) = \underline{\theta}_1$!!?!

Experiment 2

From the structure of a CCD (without center points), $\mathbf{X}'\mathbf{X}$ (based on a quadratic model without an intercept) is of form:

$$\begin{pmatrix} \mathbf{L} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B} \end{pmatrix}$$

where

$$\mathbf{L}_{r\times r} = (N_f + 2\alpha^2)\mathbf{I}, \quad \mathbf{P}_{r\times r} = 2\alpha^4\mathbf{I} + N_f\mathbf{J}, \quad \mathbf{B}_{C_2^r \times C_2^r} = N_f\mathbf{I}.$$

From this,

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{pmatrix} \mathbf{L}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}^{-1} \end{pmatrix}$$

where $\mathbf{L}^{-1}$ and $\mathbf{B}^{-1}$ are immediate. $\mathbf{P}$ is of the form $a\mathbf{I} + b\mathbf{J}$, and following the note and notation in the assignment,

$$(a\mathbf{I} + b\mathbf{J})(c\mathbf{I} + d\mathbf{J}) = ac\mathbf{I} + ad\mathbf{J} + bc\mathbf{J} + bdr\mathbf{J} = \mathbf{I}$$

or

$$ac = 1$$
$$ad + bc + brd = 0$$

for which the solution is $c = 1/a$, $d = -b/(a(a + br))$. Substituting terms into this form,

$$\mathbf{P}^{-1} = \tfrac{1}{2\alpha^4}\mathbf{I} - \tfrac{N_f}{2\alpha^4(2\alpha^4 + N_f r)}\mathbf{J}.$$

Hence, at the corner points,

$$V(\hat{y}(\underline{u})) = \sigma^2[\underline{u}'\mathbf{L}^{-1}\underline{u} + \underline{1}'\mathbf{P}^{-1}\underline{1} + \underline{x}_b'\mathbf{B}^{-1}\underline{x}_b]$$

where $\underline{x}_b$ contains the bi-linear terms $(u_i u_j, i \neq j)$,

$$= \sigma^2[\tfrac{r}{N_f + 2\alpha^2} + \tfrac{r}{2\alpha^4} - \tfrac{r^2 N_f}{2\alpha^4(2\alpha^4 + N_f r)} + \tfrac{r(r-1)}{2N_f}]$$

which can be “simplified to taste”.